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STABILITY FOR LINEARIZED GRAVITY ON THE KERR SPACETIME

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Abstract. In this paper we prove integrated energy and pointwise decay estimates for solutions of the vacuum linearized Einstein equation on the domain of outer communication of the Kerr black hole spacetime. The estimates are valid for the full subextreme range of Kerr black holes, provided integrated energy estimates for the Teukolsky Master Equation holds. For slowly rotating Kerr backgrounds, such estimates are known to hold, due to the work of one of the authors. The results in this paper thus provide the first stability results for linearized gravity on the Kerr background, in the slowly rotating case, and reduce the linearized stability problem for the full subextreme range to proving integrated energy estimates for the Teukolsky equation. This constitutes an essential step towards a proof of the black hole stability conjecture, i.e. the statement that the Kerr family is dynamically stable, one of the central open problems in general relativity.

Contents

1. Introduction

The Kerr family of asymptotically flat, stationary, and axially symmetric solutions of the vacuum Einstein equations is parametrized by mass M and angular momentum per unit mass a. In ingoing Eddington-Finkelstein coordinates[1] $(v, r, \theta, \phi) \in \mathbb{R} \times (0, \infty) \times S^2$, the Kerr metric takes the form

$$\begin{aligned} g_{ab} = & -2(dr)_{(a}(dv)_{b)} + 2a\sin^2\theta(d\phi)_{(a}(dr)_{b)} + \frac{4Mar\sin^2\theta}{\Sigma}(d\phi)_{(a}(dv)_{b)} \\ & + \frac{\Delta - a^2\sin^2\theta}{\Sigma}(dv)_a(dv)_b + \frac{a^2\sin^2\theta\Delta - (a^2+r^2)^2}{\Sigma}\sin^2\theta(d\phi)_a(d\phi)_b - \Sigma(d\theta)_a(d\theta)_b, \end{aligned} \tag{1.1}$$

with volume element $\Sigma\sin\theta dv dr d\theta d\phi$. Here $\Sigma = a^2\cos^2\theta + r^2, \Delta = a^2 - 2Mr + r^2$. The Killing vector fields of the Kerr metric are $\xi^a = (\partial_v)^a$, which has unit norm at infinity and expresses the fact that Kerr is stationary, and the axial Killing vector field $\eta^a = (\partial_\phi)^a$. In the subextreme case $|a| < M$, the maximally extended Kerr spacetime contains a black hole with a non-degenerate event horizon $\mathcal{H}$ located at $r_+ = M + \sqrt{M^2 - a^2}$, the larger of the two roots of Δ. The domain of outer communication of the Kerr black hole is the region $r > r_+$, which we shall denote $\mathcal{M}$.

In addition to being stationary and axially symmetric, the Kerr metric is algebraically special, of Petrov type D, or $\{2, 2\}$. In particular, the Weyl curvature tensor of the Kerr spacetime has two repeated principal null directions[2] l^a, n^a. We note that l^a, n^a are real, and may without loss of generality be chosen to be future directed, with n^a inward directed, $n^b\nabla_b r < 0$, and normalized so that $l^a n_a = 1$. The principal null vectors together with a pair of complex null vectors $m^a, \bar{m}^a$, where $\bar{m}^a$ is the complex conjugate of m^a, with $m^a\bar{m}_a = -1$, and perpendicular to l^a, n^a, gives a principal null tetrad $(l^a, n^a, m^a, \bar{m}^a)$. We have $g_{ab} = 2(l_{(a}n_{b)} - m_{(a}\bar{m}_{b)})$. We shall here use the Znajek tetrad [52], which in ingoing Eddinton-Finkelstein coordinates takes the form

$$l^a = \frac{\sqrt{2}a(\partial_\phi)^a}{\Sigma} + \frac{\sqrt{2}(a^2+r^2)(\partial_v)^a}{\Sigma} + \frac{\Delta(\partial_r)^a}{\sqrt{2}\Sigma}, \tag{1.2a}$$

$$n^a = -\tfrac{1}{\sqrt{2}}(\partial_r)^a, \tag{1.2b}$$

$$m^a = \frac{(\partial_\theta)^a}{\sqrt{2}(r - ia\cos\theta)} + \frac{i\csc\theta(\partial_\phi)^a}{\sqrt{2}(r - ia\cos\theta)} + \frac{ia\sin\theta(\partial_v)^a}{\sqrt{2}(r - ia\cos\theta)}, \tag{1.2c}$$

which has the useful property that n^a is auto-parallel, i.e. $n^b\nabla_b n^a = 0$. The Znajek tetrad commutes with the Killing vector fields of the Kerr spacetime and extends smoothly through the future event horizon $\mathscr{H}^+$.

[1] See [35, Box 33.2]. The ingoing Eddington-Finkelstein coordinates are also known as Kerr coordinates. We work in signature $+ - - -$, and use conventions and notations as in [39, 38].

[2] Let C_{abcd} be the Weyl tensor of $(\mathcal{M}, g_{ab})$. A null vector k^a is a principal null direction if $k_{[a}C_{a]bc[d}k_{f]}k^bk^c = 0$, cf. [44, §4.3].

Let $g_{ab}(\lambda)$ be a 1-parameter family of metrics on $\mathcal{M}$, with $g_{ab}(0) = g_{ab}$. The linearized metric $\delta g_{ab} = \frac{d}{d\lambda} g_{ab}(\lambda)\big|_{\lambda=0}$ solves the linearized vacuum Einstein equations on $\mathcal{M}$ if

$$\delta E_{ab} = 0, \tag{1.3}$$

where δE_{ab} is the linearization of the Einstein tensor at g_{ab} in the direction of δg_{ab}. Due to the covariance of Einstein's equations, the space of solutions of the linearized Einstein equation is invariant under gauge transformations

$$\delta g_{ab} \to \delta\tilde{g}_{ab} = \delta g_{ab} - 2\nabla_{(a}\nu_{b)}. \tag{1.4}$$

Upon introducing a suitable gauge condition, e.g. Lorenz gauge $\nabla^a(\delta g_{ab} - \frac{1}{2}\delta g_c{}^c g_{ab}) = 0$, the linearized Einstein equation becomes hyperbolic, and it follows from standard results that the Cauchy problem for the linearized vacuum Einstein equation on $\mathcal{M}$ admits global solutions. A priori, these may have exponential growth.

Let δg_{ab} be a solution of the linearized vacuum Einstein equation on $\mathcal{M}$, and let n^a be the ingoing principal null vector, cf. (1.2b). The fact that Kerr is of Petrov type D implies there is a vector field ν^a such that the gauge transformed metric $\delta\tilde{g}_{ab}$ satisfies [40]

$$n^b \delta\tilde{g}_{ab} = 0, \quad g^{ab}\delta\tilde{g}_{ab} = 0. \tag{1.5}$$

The resulting gauge condition is called the outgoing radiation gauge[3]. For a linearized metric in outgoing radiation gauge, the only non-vanishing components are

$$G_{00'} = \delta g_{ab} l^a l^b, \quad G_{10'} = \delta g_{ab} l^a m^b, \quad G_{20'} = \delta g_{ab} m^a m^b. \tag{1.6}$$

Let $C_{\text{hyp}} = 10^6$, and let

$$\begin{aligned} h(r) = {}& 2(r - r_+) + 4M \log\left(\frac{r}{r_+}\right) + \frac{3M^2(r_+ - r)^2}{r_+ r^2} + 2M \arctan\left(\frac{(C_{\text{hyp}} - 1)M}{r}\right) \\ & - 2M \arctan\left(\frac{(C_{\text{hyp}} - 1)M}{r_+}\right). \end{aligned} \tag{1.7}$$

Define the horizon crossing time $t_{\mathscr{H}^+}$ and the hyperboloidal time t by

$$t_{\mathscr{H}^+} = v - h/2, \tag{1.8a}$$
$$t = v - h. \tag{1.8b}$$

Then $t_{\mathscr{H}^+}$ and t are time functions with strictly spacelike levels sets which are future Cauchy surfaces in $\mathcal{M}$. The level sets of both $t_{\mathscr{H}^+}$ and t cross the event horizon and are regular there, while for large r, the level sets of $t_{\mathscr{H}^+}$ are asymptotic to spatial infinity, with an asymptotically flat induced metric, while the level sets of t are asympototic to future null infinity. Let $t_0 = 10M$ and define

$$\Sigma_{\text{init}} = \{t_{\mathscr{H}^+} = t_0\} \cap \{r > r_+\}. \tag{1.9}$$

Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$. For tensors $\varpi_{a\cdots d}$ along Σ_{init}, let $H^k_\alpha(\Sigma_{\text{init}})$ be the weighted Sobolev space with norm

$$\|\varpi\|^2_{H^k_\alpha(\Sigma_{\text{init}})} = \int_{\Sigma_{\text{init}}} M^{-\alpha} \sum_{i=0}^{k} r^{\alpha+2i-1} |\nabla^i \varpi|^2_{g_E} dr \sin\theta d\theta d\phi, \tag{1.10}$$

where the squared modulus $|\varpi|^2_{g_E}$ of a tensor is defined in terms of the positive definite metric $g_{E\,ab} = 2T_a T_b - g_{ab}$, with T^a the timelike unit normal of Σ_{init}.

Theorem 1.1. *Let $(\mathcal{M}, g_{ab})$ be the domain of outer communication of a slowly rotating Kerr spacetime, with $|a|/M \ll 1$. Let $k \in \mathbb{N}$ be sufficiently large and $\epsilon > 0$ be sufficiently small. Let δg_{ab} be a solution to the linearized vacuum Einstein equations on $(\mathcal{M}, g_{ab})$ in outgoing radiation gauge, with $\|\delta g\|_{H^k_7(\Sigma_{\text{init}})} < \infty$, and let $G_{i0'}$, $i = 0, 1, 2$ be the components of δg_{ab} defined by (1.6),*

[3]Replacing n^a by l^a leads to the ingoing radiation gauge condition. The result of [40] is valid more generally for linearized gravity on vacuum background spacetimes of Petrov type II.

with respect to the Znajek tetrad. Let $|\delta g|^2 = |G_{00'}|^2 + |G_{10'}|^2 + |G_{20'}|^2$*. There is a constant* $C = C(k, |a|/M, \epsilon)$*, such that the inequality*

$$|\delta g| \leq CM^{5/2-\epsilon} r^{-1} t^{-3/2+\epsilon} \|\delta g\|_{H^k_7(\Sigma_{\text{init}})}, \tag{1.11}$$

holds for $t > 10M$*.*

Considering the conformally rescaled metric $r^{-2}g_{ab}$ allows one to add a boundary at $r = \infty$, containing the smooth null manifolds $\mathscr{I}^+, \mathscr{I}^-$ which represent the limits of those future and past directed null geodesics, respectively, that reach infinity. The complement is spacelike infinity i_0. The future and past parts $\mathscr{H}^+, \mathscr{H}^-$, of the event horizon are reached by ingoing future and past null geodesics, respectively, that emanate in $\mathcal{M}$ and enter the black hole. The complement of $\mathscr{H}^+ \cup \mathscr{H}^-$ in $\mathscr{H}$, called the bifurcation sphere $\mathcal{B}$, is distinguished by the fact that ξ^a is tangent to $\mathcal{B}$. The coordinate v is finite on $\mathscr{H}^+$. The level sets of the horizon crossing time $t_{\mathscr{H}^+}$ are asymptotic to i_0. The level sets of t are regular at both $\mathscr{H}^+$ and $\mathscr{I}^+$, and they induce foliations of the future part of the event horizon $\mathscr{H}^+$ and future null infinity $\mathscr{I}^+$.

Let δC_{abcd} be the linearized Weyl tensor. Due to the fact that Kerr is Petrov type D, the linearized Newman-Penrose scalars

$$\vartheta\Psi_0 = -\delta C_{abcd} l^a m^b l^c m^d, \quad \vartheta\Psi_4 = -\delta C_{abcd} n^a \bar{m}^b n^c \bar{m}^d, \tag{1.12}$$

are gauge invariant. For a solution of the linearized vacuum Einstein equation on the Kerr background spacetime, $\vartheta\Psi_0, \vartheta\Psi_4$ solve a pair of decoupled wave equations called the Teukolsky Master Equations [47], and also satisfy a set of fourth-order differential relations called the Teukolsky-Starobinsky Identities [49, 43]. The linearized Einstein equations in outgoing radiation gauge reduce to the two Teukolsky Master Equations for $\vartheta\Psi_0, \vartheta\Psi_4$, the Teukolsky-Starobinsky Identities, and a set of transport equations along n^a, for the metric components (1.6) as well as for tetrad components of the linearized connection coefficients.

The compactified hyperboloidal coordinate system (t, R, θ, ϕ), with $R = 1/r$ and (θ, ϕ) as in the ingoing Eddington-Finkelstein coordinates, is regular at $\mathscr{I}^+$ considered as a null hypersurface in the conformally rescaled metric $r^{-2}g_{ab}$, as is the rescaled tetrad

$$(r^2 l^a, n^a, r m^a, r\bar{m}^a), \tag{1.13}$$

where $(l^a, n^a, m^a, \bar{m}^a)$ are given by (1.2). The asymptotic behaviours at $\mathscr{I}^+$ of tensor fields on $\mathcal{M}$, often referred to as peeling, can be understood by passing to the conformal compactification, taking into account the behaviour of the fields under conformal rescaling, and using the rescaled tetrad (1.13). A peeling analysis indicates

$$\vartheta\Psi_0 = O(r^{-5}), \qquad \vartheta\Psi_4 = O(r^{-1}), \tag{1.14a}$$

$$G_{i0'} = O(r^{-3+i}), \quad i = 0, 1, 2. \tag{1.14b}$$

The scalars $\vartheta\Psi_0, \vartheta\Psi_4$ are properly weighted in the sense of Geroch, Held, and Penrose (GHP) [24] and have boost- and spin-weights $+2, -2$, respectively. In the following, we shall transform properly weighted scalars and operators to boost-weight zero by rescaling with powers of a factor λ with boost-weight 1 and spin-weight 0, which takes the value $\lambda = 1$ in the Znajek tetrad[4]. Let

$$\hat{\psi}_{+2} = \tfrac{1}{2}(a^2 + r^2)^{1/2}(r - ia\cos\theta)^4 \lambda^{-2} \vartheta\Psi_0, \tag{1.15a}$$

$$\hat{\psi}_{-2} = \tfrac{1}{2}(a^2 + r^2)^{1/2} \lambda^2 \vartheta\Psi_4. \tag{1.15b}$$

Then $\hat{\psi}_{+2}, \hat{\psi}_{-2}$ have boost-weights 0 and spin-weights $+2, -2$, respectively. The fields $\hat{\psi}_{+2}, \hat{\psi}_{-2}$ are the de-boosted radiation fields of $\vartheta\Psi_0, \vartheta\Psi_4$, respectively, in particular they are regular, in the sense of spin-weighted fields, and non-degenerate on $\mathcal{M}$ including $\mathscr{H}^+$ and $\mathscr{I}^+$. In the following, unless otherwise stated, we shall consider only fields with boost-weight 0.

[4] Let $\rho' = \bar{m}^a m^b \nabla_b n_a$. Then ρ' is one of the GHP spin coefficients with boost-weight -1 and spin-weight 0, and $\lambda = (\sqrt{2}(r - ia\cos\theta)\rho')^{-1}$ has the desired property.

In order to discuss our estimates for the Teukolsky Master Equations, we introduce operators acting on fields of spin-weight s, which, restricting to the Znajek tetrad and the ingoing Eddington-Finkelstein coordinate system, take the explicit form

$$
\begin{aligned}
V\varphi &= \partial_v\varphi + \frac{\Delta\partial_r\varphi}{2(a^2+r^2)} + \frac{a\partial_\phi\varphi}{a^2+r^2}, && (1.16a)\\
Y\varphi &= -\partial_r\varphi, && (1.16b)\\
\mathring{\eth}\,\varphi &= \tfrac{1}{\sqrt{2}}\partial_\theta\varphi + \tfrac{i}{\sqrt{2}}\csc\theta\partial_\phi\varphi - \tfrac{1}{\sqrt{2}}s\cot\theta\varphi, && (1.16c)\\
\mathring{\eth}'\,\varphi &= \tfrac{1}{\sqrt{2}}\partial_\theta\varphi - \tfrac{i}{\sqrt{2}}\csc\theta\partial_\phi\varphi + \tfrac{1}{\sqrt{2}}s\cot\theta\varphi, && (1.16d)
\end{aligned}
$$

and

$$
\mathcal{L}_\xi\varphi = \partial_v\varphi, \qquad \mathcal{L}_\eta\varphi = \partial_\phi\varphi. \tag{1.16e}
$$

Here, V, Y represent derivatives in the directions l^a, n^a, respectively, while $\mathring{\eth}, \mathring{\eth}'$ are the spherical edth operators. Furthermore, we define V^a, Y^a to be the vector fields corresponding to V, Y. Define the operators R_s, S_s, acting on fields of spin-weight s by

$$
\begin{aligned}
R_s &= 2(a^2+r^2)YV - \frac{2a(1+2s)r}{a^2+r^2}\mathcal{L}_\eta + 4srV + \frac{2Ms(a^2-r^2)}{a^2+r^2}Y \\
&\quad + \frac{2s(M-r)r}{a^2+r^2} + \frac{\left(a^4+2Mr^3+a^2r(r-4M)\right)}{(a^2+r^2)^2}, && (1.17a)\\
S_s &= 2\,\mathring{\eth}\,\mathring{\eth}' + 2a\mathcal{L}_\xi\mathcal{L}_\eta + a^2\sin^2\theta\mathcal{L}_\xi^2 - 2ias\cos\theta\mathcal{L}_\xi. && (1.17b)
\end{aligned}
$$

If δg_{ab} solves the linearized vacuum Einstein equations, then the scalars $\hat{\psi}_s$ with $s = +2, -2$ given in (1.15) solve the Teukolsky Master Equation

$$
R_s\hat{\psi}_s - S_s\hat{\psi}_s = 0. \tag{1.18}
$$

In addition to the Teukolsky equations (1.18), the fields $\hat{\psi}_{+2}, \hat{\psi}_{-2}$ satisfy the differential identity

$$
\mathring{\eth}^4\,\hat{\psi}_{-2} = -3M\mathcal{L}_\xi(\overline{\hat{\psi}_{-2}}) - \sum_{k=1}^{4}\binom{4}{k}\mathring{\tau}^k\,\mathring{\eth}^{4-k}\,\mathcal{L}_\xi^k\hat{\psi}_{-2} + \tfrac{1}{4}\Big(Y + \frac{r}{a^2+r^2}\Big)^4\hat{\psi}_{+2}, \tag{1.19}
$$

where $\mathring{\tau} = -(r - ia\cos\theta)^2\tau$, and τ is one of the GHP spin coefficients. In the Znajek tetrad, $\mathring{\tau} = ia\sin\theta/\sqrt{2}$. Equation (1.19) is one of the Teukolsky-Starobinsky Identities, expressed in the variables (1.15) and the operators (1.16).

We introduce the set of operators

$$
\mathbb{B} = \{Y, V, r^{-1}\mathring{\eth}, r^{-1}\mathring{\eth}'\}, \tag{1.20a}
$$

related to the principal tetrad, and the set

$$
\mathbb{D} = \{MY, rV, \mathring{\eth}, \mathring{\eth}'\}, \tag{1.20b}
$$

of rescaled operators. Finally, the set

$$
\not{\mathbb{D}} = \{\mathring{\eth}, \mathring{\eth}', M\mathcal{L}_\xi\} \tag{1.20c}
$$

is appropriate for controlling fields on $\mathscr{I}^+$. In stating integral estimates, we shall make use of the volume elements

$$
\mathrm{d}^4\mu = \sin\theta dv\wedge dr\wedge d\theta\wedge d\phi, \qquad \mathrm{d}^3\mu = \sin\theta dr\wedge d\theta\wedge d\phi. \tag{1.21}
$$

Definition 1.2. Let Σ be a smooth, spacelike hypersurface, and let ν_a be a 1-form normal to Σ. Let $\mathrm{d}^3\mu_\nu$ denote a three form such that $\nu\wedge\mathrm{d}^3\mu_\nu = \mathrm{d}^4\mu$. Let φ be a boost-weight zero field. Let

k be a positive integer and define

$$E^1_\Sigma(\varphi) = M\int_\Sigma \Big((\nu_a Y^a)|V\varphi|^2 + (\nu_a V^a)|Y\varphi|^2 + (\nu_a(V^a+Y^a))r^{-2}(|\mathring{\eth}\,\varphi|^2 + |\mathring{\eth}'\,\varphi|^2)\Big)\,\mathrm{d}^3\mu_\nu, \tag{1.22a}$$

$$E^k_\Sigma(\varphi) = \sum_{i=0}^{k-1}\sum_{X_1,\dots,X_i\in\mathbb{B}} M^{2i}E^1_\Sigma(X_i\dots X_1\varphi), \tag{1.22b}$$

$$B^1_{t_1,t_2}(\varphi) = \int_{\Omega_{t_1,t_2}\cap\{r\geq 10M\}} M^3r^{-3}\sum_{X\in\mathbb{B}}|X\varphi|^2\mathrm{d}^4\mu + \int_{\Omega_{t_1,t_2}} Mr^{-3}|\varphi|^2\mathrm{d}^4\mu, \tag{1.22c}$$

$$B^k_{t_1,t_2}(\varphi) = \sum_{i=0}^{k-1}\sum_{X_1,\dots,X_i\in\mathbb{B}} M^{2i}W^1_{t_1,t_2}(X_i\dots X_1\varphi). \tag{1.22d}$$

In order to discuss our second main result, we shall need the fields

$$\hat\psi^{(i)}_{-2} = \left(\frac{a^2+r^2}{M}V\right)^i\hat\psi_{-2}, \qquad 0\leq i\leq 4, \tag{1.23}$$

defined in terms of derivatives of $\hat\psi_{-2}$. Let Σ_t be a level set of the hyperboloidal time function t, cf. (1.8b). For $t_1<t_2$, let Ω_{t_1,t_2} denote the spacetime domain given by the intersection of the past of Σ_{t_2}, with the future of Σ_{t_1}.

Definition 1.3 (Basic decay condition)**.**
Let δg_{ab} be a solution to the linearized Einstein equations on the domain of outer communication $\mathcal{M}$ of a Kerr black hole spacetime, and let $\hat\psi_{+2}$ be as in (1.15a), and let $\hat\psi^{(i)}_{-2}$, $i=0,1,2$ be as in (1.23). We shall say that δg_{ab} satisfies the basic decay condition if the following holds for all sufficiently large $k\in\mathbb{N}$.

(1) There is a positive constant C such that for all $t_1<t_2$ with $10M\leq t_1$,

$$\sum_{i=0}^{2}\Big(E^k_{\Sigma_{t_2}}(\hat\psi^{(i)}_{-2}) + B^k_{t_1,t_2}(\hat\psi^{(i)}_{-2})\Big) \leq C\sum_{i=0}^{2}E^k_{\Sigma_{t_1}}(\hat\psi^{(i)}_{-2}), \tag{1.24}$$

(2)

$$\lim_{t\to\pm\infty}\Big(|\hat\psi_{+2}|_{k,\not{D}}|_{\mathscr{I}^+}\Big) = 0. \tag{1.25}$$

Remark 1.4. The spin-weight -2 case, point 1, of definition 1.3 is an integrated energy estimate. The spin-weight $+2$ condition in point 2, on the other hand, is not in the form of an estimate, but rather a weak pointwise decay condition. In section 7, equation (1.25) is proved to follow from a basic integrated energy estimate analogous to the condition stated in inequality (1.24).

We are now able to formulate the second main result of this paper.

Theorem 1.5. *Let $(\mathcal{M}, g_{ab})$ be the domain of outer communication of a subextreme Kerr spacetime. Let $k\in\mathbb{N}$ be sufficiently large and $\epsilon>0$ be sufficiently small. Let δg_{ab} be a solution to the linearized vacuum Einstein equations on $(\mathcal{M}, g_{ab})$ in outgoing radiation gauge, with $\|\delta g\|_{H^k_7(\Sigma_{\mathrm{init}})}<\infty$, and let $G_{i0'}$, $i=0,1,2$ be the components of δg_{ab} defined by (1.6), with respect to the Znajek tetrad.*

Assume that δg_{ab} satisfies the basic decay conditions of definition 1.3. Then, there is a constant $C=C(k,|a|/M,\epsilon)$, such that the following inequalities hold for $t>10M$.

(1) In the interior region $r<t$,

$$|G_{20'}| \leq Cr^{-1}t^{-5/2+\epsilon}\|\delta g\|_{H^k_7(\Sigma_{\mathrm{init}})}, \tag{1.26a}$$

$$|G_{i0'}| \leq Cr^{-2}t^{-3/2+\epsilon}\|\delta g\|_{H^k_7(\Sigma_{\mathrm{init}})}, \quad \text{for } i\in\{0,1\}. \tag{1.26b}$$

(2) In the exterior region $r\geq t$,

$$|G_{i0'}| \leq Cr^{i-3}t^{-i-1/2+\epsilon}\|\delta g\|_{H^k_7(\Sigma_{\mathrm{init}})}, \quad \text{for } i\in\{0,1,2\}. \tag{1.27}$$

Remark 1.6. (1) It follows from the work in [32] together with the arguments in section 7 that the conditions stated in definition 1.3 hold for a solution δg_{ab} of the linearized Einstein equation on a slowly rotating Kerr background, with $\|\delta g\|_{H^k_7(\Sigma_{\text{init}})} < \infty$.

(2) Theorem 1.5 is valid for the whole subextreme range $|a| < M$, provided that the basic decay condition, definition 1.3 holds.

(3) As part of the proof of theorems 1.1 and 1.5 we prove decay estimates for $\hat{\psi}_{-2}$ which are stronger than those previously available.

(4) The fall-off at $\mathscr{I}^+$, with respect to r, for the metric components $G_{i0'}$, that is expressed in the inequality (1.27), is compatible to that predicted by a peeling analysis for δg_{ab}.

(5) The linearized mass δM and linearized angular momentum per unit mass δa can be evaluated by linearized Komar integrals over spheres in $\mathcal{M}$. For linearized gravity on the Kerr background, the Komar integrals define conserved charges depending only on the topological class of the sphere [1]. The fall-off conditions on initial data in theorems 1.1 and 1.5 imply that $\delta M = \delta a = 0$.

The first main step in the proof of theorem 1.5 is to convert the basic energy and Morawetz estimates of definition 1.3 into strong energy and pointwise decay estimates for the Teukolsky scalars. After a suitable rescaling, the higher order de-boosted Teukolsky scalars with spin-weight -2, $\hat{\psi}^{(i)}_{-2}$, $i = 0, \dots, 4$, defined in (1.23), solve a 5×5 system of spin-weighted wave equations. The right-hand side of this system has only first order derivatives, involving V and $\mathcal{L}_\eta$. Using a weighted multiplier estimate with the multiplier $r^\alpha V$, for $0 < \alpha < 2$, applied sequentially to this system, the assumed basic decay conditions imply a hierachy of integrated energy estimates for weighted energies of the form

$$\|\varphi\|^2_{W^k_\alpha(\Sigma_t)} = \sum_{i=0}^{k} \sum_{X_1,\dots,X_i \in \mathbb{D}} \int_{\Sigma_t} M^{-\alpha-1} r^\alpha |X_1 \cdots X_i \varphi|^2 \mathrm{d}^3\mu, \tag{1.28}$$

which via the pigeonhole principle can be converted to time decay estimates. Here it is important that the angular part of the spin-weighted wave equation under consideration has either a positive lower bound on the spectrum for the angular operator. This consideration constrains the size of the derived system, the length of the hierarchy of weighted estimates, and consequently the fall-off rates provided by the estimates.

The second main step is to derive, from the linearized Einstein equations in outgoing radiation gauge, a set of transport equations, each of which is of the form

$$Y\varphi = \varrho, \tag{1.29}$$

for a set of fields, which includes de-boosted and rescaled versions of the metric components $G_{i0'}$, as well as fields derived from the linearized connection coefficients. The solution is determined by $\hat{\psi}_{-2}$, as well as the initial data given on Σ_{init}. The solutions of the hierarchy of transport equations are estimated using weighted Hardy estimates, which yield integrated, weighted energy estimates for the fields in the hierarchy starting from the integrated, weighted energy estimates for $\hat{\psi}_{-2}$. A subtlety here is that the weighted Hardy estimates apply only to fields with sufficient fall-off at $\mathscr{I}^+$. This makes it necessary to consider Taylor expansions at $\mathscr{I}^+$ and to treat the Taylor coefficients on $\mathscr{I}^+$ separately from the remainder terms. The Taylor coefficients on $\mathscr{I}^+$ satisfy a set of transport equations, which can be integrated due to the Teukolsky-Starobinsky Identity and the condition (1.25). In performing these estimates, it turns out to be important to treat the exterior region $r \geq t$ separately from the interior region $r < t$. The transport estimates from the exterior region provide decay estimates on the transition region $r = t$. These provide part of the source for the estimates in the interior region.

We shall now put the results presented here in context and give some background and references. The Kerr [28] family of stationary and rotating black hole solutions to the vacuum Einstein equation is conjectured to be unique and dynamically stable, and a proof of these conjectures is required to establish the validity of the Kerr black hole as a physical model. The black hole uniqueness conjecture states that any asympotically flat, stationary, vacuum spacetime containing a non-degenerate black hole is isometric to a member of the subextreme Kerr family. See [13] for a recent review on the black hole uniqueness problem. The black hole stability conjecture, on the other hand, states that the maximal Cauchy development of data for the vacuum Einstein equation that is close, in a suitable sense, to Kerr data, is asymptotic at timelike infinity to a member of the Kerr family. The mathematical problems resulting from the uniqueness and stability conjectures have stimulated much work during the last five decades, but, in spite of significant progress, both the stability and uniqueness conjectures remain open.

There are important similarities between the stability problem for Minkowski space and the black hole stability problem, and the ideas and techniques introduced in the work on this problem have had a significant influence in work on the black hole stability problem. In particular, we mention the approach based on conformal compactification used by Friedrich [23] in his proof of the future stability of Minkowski space, and the vector-fields based energy estimates used in the monumental proof of the non-linear stability of Minowski space of Christodoulou and Klainerman [12].

The linearized counterpart to the black hole stability conjecture is the statement that a solution to the linearized Einstein equations, in a suitable gauge, generated from initial data that is well-behaved at spatial infinity, tends at timelike infinity to a linearized perturbation of the Kerr background with respect to the moduli degrees of freedom of the Kerr famility, i.e. mass and angular momentum per unit mass. Following nearly two decades work on decay estimates for solutions of wave equations (spin-0) and Maxwell fields (spin-1) on Schwarzschild and Kerr backgrounds, cf. [20, 17, 45, 4, 5, 46, 34] and references therein, the first such results for linearized gravity on the Schwarzschild background [14, 27] have appeared, see also [6, 18]. The technique introduced in [8], see also [41, 36], was influential in our approach to treating the $\hat{\psi}^{(i)}_{-2}$. Recently, decay estimates for the Teukolsky equations in the spin-2 case on slowly rotating Kerr backgrounds [32, 15] have been proved, see also [21].

Energy and Morawetz estimates on the full subextreme range of Kerr backgrounds is known at present only for the spin-0 case [17]. Here, Whiting's mode-stability result [51] and its generalization to real frequencies in the spin-0 case [42] play a central role. The results mentioned above for fields with non-zero spin provide energy and integrated energy estimates for the Teukolsky equation in the slowly rotating case. It can be expected that known techniques based on the generalization of Whiting's mode stability result to the real frequency case, with non-zero spin [7], will yield a proof of the corresponding estimates for the full subextreme range of Kerr black holes.

There has recently been significant progress on the non-linear black hole stability problem. For the case of of slowly rotating Kerr-de Sitter black holes, non-linear stability is known [26]. The presence of a positive cosmological constant in the Kerr-de Sitter case provides exponential fall-off in time, which plays an important role in the proof. The non-linear stability of the Schwarzschild spacetime with respect to polarized axially symmetric perturbations is also known [29]. In particular, the assumptions in the just cited paper imply that the spacetime geometry is asymptotic at timelike infinity, to a Schwarzchild spacetime.

In the present paper we give the first proof of linearized stability of the Kerr black hole, by providing energy bounds, Morawetz estimates, and pointwise decay estimates for the linearized vacuum Einstein equation on the domain of outer communication of a slowly rotating Kerr black hole, see theorem 1.1. Our second main result, theorem 1.5, also implies the linear stability in the full subextreme range $|a| < M$ subject to the basic decay condition stated in definition 1.3. The estimates represent a major step towards a proof of the non-linear stability of the Kerr spacetime, without the additional assumption of axial symmetry.

Overview of this paper. In section 2, we collect the geometric preliminaries needed in the paper. These include the hyperboloidal time function as well as a discussion of the operators on spin-weighted fields that will be used in the remainder of the paper. The linearized Einstein equation is presented in section 3.1. In section 3.2, the outgoing radiation gauge condition is discussed. The main transport system, resulting from the linearized Einstein equations in outgoing radiation gauge, and which relates the spin-weight -2 Teukolsky scalar to the linearized connection coefficients and the linearized metric, is presented in section 3.3. We also state, in sections 3.4 and 3.5, the Teukolsky Master Equations and one of the Teukolsky-Starobinsky Identities in the form that we use.

Section 4 collects some analysis results needed in the main part of the paper. These include definitions of norms and basic estimates for spin-weighted operators. Further, three important lemmas, which play a central role in the decay estimates, are developed in section 5. The first is a basic lemma which can be used to convert weighted energy estimates to energy decay estimates. This type of result is often referred to as the pigeonhole principle. The second is a weighted Hardy estimate for transport equations, and the third is a weighted multiplier estimate for the spin-weighted wave equation.

Sections 6 and 7 present the decay estimates for the Teukolsky scalars. The estimates presented here assume a basic integrated energy decay estimate, but they do not require slow rotation, i.e. smallness of $|a|/M$.

Finally, in section 8, we use the transport system derived in section 3.3, and the decay estimates proved in sections 6 and 7, to prove estimates for linearized connection coefficents and metric components. The method used here involves the analysis of Taylor expansions at $\mathscr{I}^+$. Taylor coefficents are shown to satisfy transport equations on $\mathscr{I}^+$, and decay for these is proved using a Teukolsky-Starobinsky identity. The Taylor remainder is controlled using the estimates proved in section 5. This section ends with the proof of the main theorems.

There are two appendices. Appendix A provides the complete first-order form of the linearized Einstein equations that is the basis for all computations in the paper. These are also presented in a form specialized to outgoing radiation gauge. Finally, appendix B provides information on the form of the non-radiating solutions to the linearized Einstein equations in outgoing radiation gauge on the Kerr background that represent linearized mass and angular momentum.

2. Geometric preliminaries

2.1. **Notation and conventions.** We shall use index and sign conventions following Penrose and Rindler [39, 38], see also [3] for background. We work with tensors and 2-spinors using abstract indices, and make use of scalar components of tensors and spinors, defined by projecting on a null tetrad. The resulting scalars are properly weighted in the sense of Geroch, Held and Penrose (GHP), cf. [24]. The GHP formalism provides a covariant framework which is convenient for calculations. In particular, we shall use the GHP operators $þ, þ', ð, ð'$, corresponding to derivatives along tetrad legs. Unless otherwise stated, we shall assume that all fields are properly weighted and smooth, in the appropriate sense.

The first-order system of transport equations which is used here, cf. sections 3.1, 3.3 and appendix A, has been derived using the covariant formalism for calculus of variations with spinors introduced by Bäckdahl and Valiente-Kroon in [11] and is closely related to the first order form of the Einstein equations as a system of scalar equations derived by Penrose and collaborators in [24] and [37]. The computer algebra tools for calculations in the 2-spinor and GHP formalisms developed by Aksteiner and Bäckdahl [9, 10], and related packages, have played a central role in deriving the equations used in this paper.

An oriented 3+1 dimensional globally hyperbolic spacetime, such as the domain of outer communication of the Kerr black hole spacetime, is a spin manifold. The spin group in this case is $\mathrm{SL}(2,\mathbb{C})$, the double cover of $SO_0(1,3)$. The spinor space is $\mathbb{C}^2$ with the vector representation of $\mathrm{SL}(2,\mathbb{C})$ and the complex conjugate representation is denoted $\overline{\mathbb{C}}^2$. Sections of the spinor bundles associated to $\mathbb{C}^2$ and $\overline{\mathbb{C}}^2$ are denoted with capital latin indices, and primed capital latin indices, respectively. The term spinor is used for sections of these bundles as well as of their tensor products, eg. $\varphi_{A\cdots DA'\cdots D'}$.

The isomorphism $\mathbb{C}^2 \otimes \overline{\mathbb{C}}^2 \approx \mathbb{R}^4 \otimes \mathbb{C}$ provides a correspondence between vectors and spinors, which extends to a correspondence between tensors and spinors, expressed via the soldering form $g_a{}^{AA'}$, eg. $\nu_a = g_a{}^{AA'}\nu_{AA'}$. It is convenient to write this correspondence in the abbreviated form $\nu_a = \nu_{AA'}$. The action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$ leaves an area element $\epsilon_{AB} = \epsilon_{[AB]}$ invariant. If the normalization $g_{ab} = \epsilon_{AB}\bar{\epsilon}_{A'B'}$ defines the spin metric ϵ_{AB} up to a phase. This is used to raise and lower spinor indices.

Using the tensor-spinor correspondence mentioned above, it is possible to express any tensor as a sum of symmetric spinors multiplied by ϵ_{AB} factors. For the Weyl tensor, we have

$$C_{abcd} = \Psi_{ABCD}\bar{\epsilon}_{A'B'}\bar{\epsilon}_{C'D'} + \epsilon_{AB}\epsilon_{CD}\bar{\Psi}_{A'B'C'D'}, \tag{2.1}$$

where Ψ_{ABCD} is the symmetric Weyl spinor.

Let a spin dyad o_A, ι_A, i.e. a local frame for the spinor bundle, with the normalization
$$o_A \iota^A = 1 \tag{2.2}$$
be given. For a symmetric spinor $\varphi_{A\cdots DA'\cdots D'}$, scalar components $\varphi_{ii'}$ are defined by contracting i times with ι^A, i' times with $\iota^{A'}$, and contracting the remaining indices with o^A or $o^{A'}$. The numbers i or i' are omitted if the spinor is of valence $(0,l)$ or $(k,0)$ respectively. In particular, the Weyl spinor Ψ_{ABCD} corresponds to the five complex Weyl scalars $\Psi_i, i = 0, \dots, 4$.

A complex null tetrad is given in terms of the spin dyad by
$$l^a = o^A o^{A'}, \quad n^a = \iota^A \iota^{A'}, \quad m^a = o^A \iota^{A'}, \quad \bar{m}^a = \iota^A o^{A'}. \tag{2.3}$$
Here $l^a, n^a, m^a, \bar{m}^a$ are null, l^a, n^a being real, and $m^a, \bar{m}^a$ complex, with $\bar{m}^a$ the complex conjugate of m^a. The null tetrad satisfies the normalization $l^a n_a = -m^a \bar{m}_a = 1$, and hence
$$g_{ab} = 2(l_{(a} n_{b)} - m_{(a} \bar{m}_{b)}). \tag{2.4}$$
We only consider situations where there is a pair of globally defined real null directions and restrict attention to dyads such that the real null legs of the tetrad induced by (2.3) are parallel to these. Typically there is no global choice of the complex null vectors $m^a, \bar{m}^a$, nor dyad elements o_A, ι_A.

The normalization (2.2) is left invariant by rescalings $o_A \to \lambda o_A$, $\iota_A \to \lambda^{-1} \iota_A$ where $\lambda \neq 0$ is a complex scalar field. Scalar fields φ defined by projecting spinors on the dyad, or tensors on the tetrad, transform as $\varphi \to \lambda^p \bar{\lambda}^q \varphi$, for integers p, q. Such scalars are called properly weighted, with type $\{p, q\}$. An example is given by the component $\nu_a n^a = \nu_{AA'} \iota^A \iota^{A'}$, for a vector field ν^a, which has type $\{-1,-1\}$. Using the notation introduced above, this would be denoted $\nu_{11'}$. The boost weight of a properly weighted scalar of type $\{p, q\}$ is $b = (p+q)/2$ and the spin weight is $s = (p-q)/2$. The notions of properly weighted scalar, type, as well as boost- and spin-weight extend to tensor and spinor fields. For example, m^a has type $\{1,-1\}$, boost-weight 0, and spin-weight 1. A field of GHP type $\{0,0\}$ is well-defined, independent of rescalings of the tetrad. Examples are the metric g_{ab} and the middle Weyl scalar $\Psi_2 = \Psi_{ABCD} o^A o^B \iota^C \iota^D$.

Computations using the GHP formalism are simplified by using the prime and complex conjugation operations[5] . Complex conjugation, $\varphi \to \bar{\varphi}$ takes fields of type $\{p, q\}$ to type $\{q, p\}$, i.e. it changes the sign of the spin-weight, and preserves the boost-weight. The prime operation, $\varphi \to \varphi'$, interchanges $l^a \leftrightarrow n^a$, $m^a \leftrightarrow \bar{m}^a$, and takes fields of type $\{p, q\}$ to fields of type $\{-p, -q\}$. The prime operation and complex conjugation commute and are symmetries in the sense that an equation valid in the GHP formalism remains valid after applying the prime operation or complex conjugation. The GHP type and the boost- and spin-weights are additive under multiplication.

Properly weighted scalars are sections of complex line bundles, and more generally, properly weighted tensor and spinor fields are sections of complex vector bundles. The lift of the Levi-Civita connection ∇_a to these bundles gives a covariant derivative denoted Θ_a. Projecting on the null tetrad $l^a, n^a, m^a, \bar{m}^a$ gives the GHP operators [24],
$$\text{þ} = l^a \Theta_a, \quad \text{þ}' = n^a \Theta_a, \quad \eth = m^a \Theta_a, \quad \eth' = \bar{m}^a \Theta_a.$$
See [25] for discussion of the geometry of properly weighted scalars and the GHP covariant derivative. The GHP operators are properly weighted, in the sense that they take properly weighted fields to properly weighted fields, for example if φ has type $\{p, q\}$, then $\text{þ}\varphi$ has type $\{p+1, q+1\}$. This can be seen from the fact that $l^a = o^A \bar{o}^{A'}$ has type $\{1, 1\}$.

There are twelve connection coefficients in a null frame, up to complex conjugation. Of these, eight are properly weighted, and are given by
$$\kappa = m^b l^a \nabla_a l_b, \quad \sigma = m^b m^a \nabla_a l_b, \quad \rho = m^b \bar{m}^a \nabla_a l_b, \quad \tau = m^b n^a \nabla_a l_b, \tag{2.5}$$
together with their primes $\kappa', \sigma', \rho', \tau'$. These are the GHP spin coefficients. The remaining four connection coefficients, given by
$$\epsilon = \frac{1}{2}(n^a l^b \nabla_b l_a + m^a l^b \nabla_b \bar{m}_a) \quad \beta = \frac{1}{2}(n^a m^b \nabla_b l_a + m^a m^b \nabla_b \bar{m}_a) \tag{2.6}$$
and their primes, enter in the connection 1-form for the connection Θ_a. We have
$$\Theta_a \varphi = \nabla_a \varphi - b n^b \nabla_a l_b \varphi + s \bar{m}^b \nabla_a m_b \varphi \tag{2.7}$$

[5]In addition, there is the Sachs $*$ operation, see [24].

where φ is a properly weighted scalar with boost- and spin-weight b, s. This also extends to properly weighted tensor and spinor fields.

Remark 2.1. Let φ be a properly weighted scalar with boost-weight zero and spin-weight s. By the above, we have that $\bar{\varphi}$ has spin-weight $-s$, and hence

$$|\varphi|^2 = \varphi\bar{\varphi} \tag{2.8}$$

is a true scalar with GHP type $\{0, 0\}$. Introducing the inner product

$$\langle\varphi, \varrho\rangle = \varphi\bar{\varrho}, \tag{2.9}$$

we may view spin-weighted fields as sections of a Riemannian vector bundle. The GHP covariant derivative Θ_a is real, in the sense that

$$\Theta_a\bar{\varphi} = \overline{\Theta_a\varphi} \tag{2.10}$$

and hence it is also metric, with respect to the inner product given by (2.9),

$$\nabla_a\langle\varphi, \varrho\rangle = \langle\Theta_a\varphi, \varrho\rangle + \langle\varphi, \Theta_a\varrho\rangle. \tag{2.11}$$

2.2. **Geometry of Kerr.** The Kerr spacetime is of Petrov type D, or $\{2, 2\}$, which means that one can find a prinicipal dyad o_A, ι_A such that

$$\Psi_{ABCD} = 6\Psi_2 o_{(A}o_B\iota_C\iota_{D)}. \tag{2.12}$$

The corresponding tetrad, defined by (2.3), is called a principal tetrad, and the real null vectors l^a, n^a give the principal null directions of the Weyl tensor. The main feature of the Petrov type D geometry is encoded in the symmetric Killing spinor κ_{AB} found in [50], satisfying

$$\nabla_{(A}{}^{A'}\kappa_{BC)} = 0. \tag{2.13}$$

In a principal dyad the Killing spinor takes the simple form

$$\kappa_{AB} = -2\kappa_1 o_{(A}\iota_{B)}. \tag{2.14}$$

Note that κ_1 and Ψ_2 can be expressed covariantly via the relations $\kappa_{AB}\kappa^{AB} = -2\kappa_1{}^2$ and $\Psi_{ABCD}\Psi^{ABCD} = 6\Psi_2^2$. Hence, we can allow κ_1 and Ψ_2 in covariant expressions. In the Kerr spacetime, κ_{AB} can be normalized so that the stationary Killing field with unit norm at infinity is given by

$$\xi_{AA'} = \nabla^B{}_{A'}\kappa_{AB}. \tag{2.15}$$

The Eddington-Finkelstein (or Boyer-Lindquist) coordinates r, θ can be defined covariantly via

$$r = -\tfrac{3}{2}(\kappa_1 + \bar{\kappa}_{1'}), \tag{2.16a}$$

$$a\cos\theta = -\tfrac{3}{2}i(\kappa_1 - \bar{\kappa}_{1'}). \tag{2.16b}$$

The geometric definition of the radial coordinate r remains valid in the non-rotating case, $a = 0$. Similarly, we define the boost- and spin-weight zero quantities

$$\Delta = -162\kappa_1{}^3\bar{\kappa}_{1'}\rho\rho', \qquad \Sigma = 9\kappa_1\bar{\kappa}_{1'}. \tag{2.17}$$

In a principal tetrad this corresponds to the standard Δ and Σ, which take the form

$$\Sigma = a^2\cos^2\theta + r^2, \qquad \Delta = a^2 - 2Mr + r^2. \tag{2.18}$$

The remaining two coordinates can be chosen to correspond to the two Killing fields of the spacetime.

We now give the concrete coordinate form of the Kerr metric in the ingoing Kerr coordinates (v, r, θ, ϕ), cf. [35, Box 33.2]. In the Schwarzschild case, the ingoing Kerr coordinates coincide with the ingoing Eddington-Finkelstein (IEF) coordinates, and we shall here use that term also in the Kerr case. The principal Znajek [52] tetrad in IEF coordinates takes the form

$$l^a = \frac{\sqrt{2}a(\partial_\phi)^a}{\Sigma} + \frac{\sqrt{2}(a^2 + r^2)(\partial_v)^a}{\Sigma} + \frac{\Delta(\partial_r)^a}{\sqrt{2}\Sigma}, \tag{2.19a}$$

$$n^a = -\tfrac{1}{\sqrt{2}}(\partial_r)^a, \tag{2.19b}$$

$$m^a = \frac{(\partial_\theta)^a}{\sqrt{2}(r - ia\cos\theta)} + \frac{i\csc\theta(\partial_\phi)^a}{\sqrt{2}(r - ia\cos\theta)} + \frac{ia\sin\theta(\partial_v)^a}{\sqrt{2}(r - ia\cos\theta)}, \tag{2.19c}$$

with $\bar{m}^a$ the complex conjugate of m^a. The Kerr metric in IEF coordinates can be written using (2.4) as

$$\begin{aligned} g_{ab} = &-2(dr)_{(a}(dv)_{b)} + 2a\sin^2\theta(d\phi)_{(a}(dr)_{b)} + \frac{4Mar\sin^2\theta}{\Sigma}(d\phi)_{(a}(dv)_{b)} \\ &+ \frac{\Delta - a^2\sin^2\theta}{\Sigma}(dv)_a(dv)_b + \frac{a^2\sin^2\theta\Delta - (a^2+r^2)^2}{\Sigma}\sin^2\theta(d\phi)_a(d\phi)_b - \Sigma(d\theta)_a(d\theta)_b. \end{aligned} \tag{2.20}$$

The volume element of g_{ab} is given by

$$\sin\theta\Sigma dv dr d\theta d\phi. \tag{2.21}$$

We shall often use ω to denote the angular coordinates (θ, ϕ).

Remark 2.2. The Killing vector fields

$$\xi^a = (\partial_v)^a, \qquad \zeta^a = a^2(\partial_v)^a + a(\partial_\phi)^a, \qquad \eta^a = a^{-1}\zeta^a - a\xi^a = (\partial_\phi)^a, \tag{2.22}$$

are naturally defined in terms of the Killing spinor, cf. [3], provided $a \neq 0$. The form of ξ^a given here agrees with (2.15). In the Schwarzschild case $a = 0$, defining an azimuthal vector field $\eta^a = (\partial_\phi)^a$ corresponds to a choice of rotation axis.

Remark 2.3. The only non-vanishing components, in a principal dyad, of the Killing spinor and curvature in the Kerr spacetime are, in IEF coordinates,

$$\kappa_1 = -\tfrac{1}{3}(r - ia\cos\theta), \qquad \Psi_2 = -M(r - ia\cos\theta)^{-3}. \tag{2.23}$$

Remark 2.4. (1) The spin coefficients in the Znajek tetrad (2.19) are

$$\kappa = 0, \qquad \kappa' = 0, \qquad \sigma = 0, \qquad \sigma' = 0, \tag{2.24a}$$

$$\rho = \frac{\Delta}{3\sqrt{2}\kappa_1\Sigma}, \qquad \rho' = -\frac{1}{3\sqrt{2}\kappa_1}, \qquad \tau = -\frac{ia\sin\theta}{9\sqrt{2}\kappa_1{}^2}, \qquad \tau' = -\frac{ia\sin\theta}{\sqrt{2}\Sigma}, \tag{2.24b}$$

$$\epsilon' = 0, \qquad \beta' = -\frac{\cot\theta}{6\sqrt{2}\bar{\kappa}_{1'}}, \qquad \beta = -\frac{i\csc\theta(2a - 3i\cos\theta\bar{\kappa}_{1'})}{18\sqrt{2}\kappa_1{}^2}, \tag{2.24c}$$

$$\epsilon = \frac{2\Delta - 6M\kappa_1 - 9\kappa_1{}^2 - \Sigma}{6\sqrt{2}\kappa_1\Sigma}. \tag{2.24d}$$

(2) Due to the fact that $\epsilon' = 0$, the ingoing null leg n^a is auto-parallel

$$n^b\nabla_b n^a = 0, \tag{2.25}$$

i.e. it generates affinely parametrized geodesics.

Remark 2.5. We make use of the covariant GHP formalism and properly weighted scalars, and hence our calculations are independent of the specific coordinate system and principal tetrad used. However, it is sometimes convenient to make use of the ingoing Eddington-Finkelstein coordinate system and the explicit form of the Znajek tetrad.

In the subextreme case $|a| < M$, the event horizon $\mathscr{H}$ of the Kerr black hole is located at $r = r_+$, where

$$r_+ = M + \sqrt{M^2 - a^2} \tag{2.26}$$

is the largest root of Δ. The exterior, or domain of outer communication, of the Kerr black hole, $(\mathcal{M}, g_{ab})$ is, in ingoing Eddington-Finkelstein coordinates, represented by $(v, r, \theta, \phi) \in \mathbb{R} \times (r_+, \infty) \times S^2$. The event horizon is the boundary of the domain of outer communication.

Past- and future-directed causal geodesics that start in $\mathcal{M}$ and fall into the black hole cross the past and future parts, $\mathscr{H}^-, \mathscr{H}^+$, of the event horizon, respectively. The complement of $\mathscr{H}^+ \cup \mathscr{H}^-$ in $\mathscr{H}$ is the bifurcation sphere $\mathcal{B}$.

Definition 2.6. (1) The tortoise coordinate $r_* = r_*(r)$ is defined by

$$\frac{dr_*}{dr} = \frac{r^2 + a^2}{\Delta}, \quad r_*(3M) = 0. \tag{2.27}$$

Further, let $r^\sharp = r^\sharp(r)$ be defined by

$$dr^\sharp = \frac{a}{\Delta}, \quad r^\sharp(3M) = 0. \tag{2.28}$$

(2) The Boyer-Lindquist time t_{BL} is

$$t_{BL} = v - r_*. \tag{2.29}$$

Let $\phi_{BL} = \phi - r^\sharp$. The Boyer-Lindquist coordinate system is given by $(t_{BL}, r, \theta, \phi_{BL})$.

(3) The retarded time u is

$$u = v - 2r_*. \tag{2.30}$$

The outgoing Kerr, or Eddington-Finkelstein coordinates are $(u, r, \theta, \phi^\sharp)$ where the modified azimuthal angle is defined by $\phi^\sharp = \phi - 2r^\sharp$.

Remark 2.7. We shall sometimes refer to v as the advanced time. However, neither u nor v is a time function, in particular their level sets are weakly timelike, in the non-static Kerr case $(a \neq 0)$.

For later use, we note that letting $R = 1/r$, the conformal rescaling $R^2 g_{ab}$ allows adding a conformal boundary at $R = 0$. Future and past null infinity $\mathscr{I}^+, \mathscr{I}^-$ represent the set of endpoints of future and past directed outgoing null geodesics that escape from $\mathcal{M}$, respectively. The complement of $\mathscr{I}^- \cup \mathscr{I}^+$ in the conformal boundary is called spacelike infinity, and denoted i_0. Finally, we denote future and past timelike infinity, which represent the asymptotic future and past of causal curves in $\mathcal{M}$ that neither escape through $\mathscr{I}$ nor fall through $\mathscr{H}$, by i_+ and i_-, respectively.

The compactified outgoing coordinates $(u, R, \theta, \phi^\sharp)$ extend to future null infinity $\mathscr{I}^+$, and cover the past horizon $\mathscr{H}^-$. Similarly, the compactified ingoing coordinate system (v, R, θ, ϕ) covers past null infinity $\mathscr{I}^-$ and the future horizon $\mathscr{H}^+$. Level sets of the Boyer-Lindquist time t_{BL} reach the bifurcation sphere $\mathcal{B}$ at $r = r_+$ and spacelike infinity i_0 as $r \to \infty$. While the conformally rescaled metric $R^2 g_{ab}$ is regular at $\mathscr{I}^- \cup \mathscr{I}^+$, which are smooth null surfaces, it fails to be regular at i_0. See figure 1.

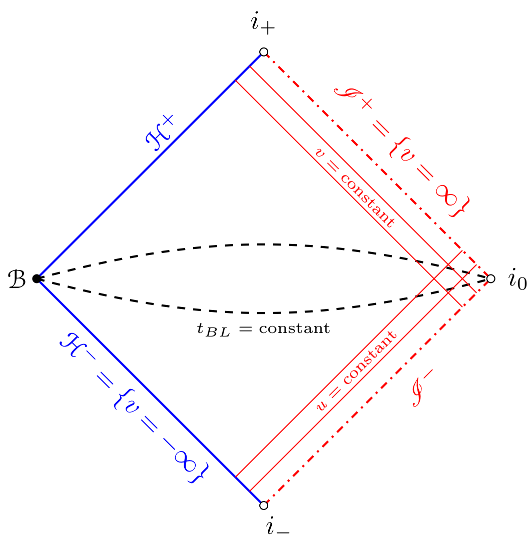

FIGURE 1. The Kerr DOC, with t_{BL}, v, and u level sets indicated.

2.3. Operators on spin-weighted scalars.

Definition 2.8. (1) A properly weighted scalar of boost-weight zero is called a spin-weighted scalar. Unless otherwise specified, the spin-weight is denoted by s.

(2) A properly weighted operator of boost-weight zero is called a spin-weighted operator.

Definition 2.9. Define

$$\lambda = (-3\sqrt{2}\kappa_1\rho')^{-1}. \tag{2.31}$$

Remark 2.10. The spin coefficient ρ' is properly weighted with boost-weight -1 and spin-weight 0. The scalar λ defined in (2.31) has boost-weight 1, spin-weight 0 and takes the value 1 in the Znajek tetrad. By multiplying with powers of ρ' or λ we may de-boost operators and scalars, so that they have boost weight zero. We shall apply this operation systematically.

Definition 2.11. Let λ be as in definition 2.9.

(1) Define the following spin-weighted operators

$$V\varphi = \frac{\Sigma}{\sqrt{2}\lambda(a^2+r^2)}\,\text{þ}\,\varphi + \frac{27s{\kappa_1}^2(\kappa_1-\bar{\kappa}_{1'})\rho\rho'\varphi}{a^2+r^2}, \tag{2.32a}$$
$$Y\varphi = \sqrt{2}\lambda\,\text{þ}', \tag{2.32b}$$
$$\mathring{\eth}\,\varphi = -9\mathcal{L}_\xi\varphi{\kappa_1}^2\tau + 3s\kappa_1\tau\varphi + 3\kappa_1\,\eth\,\varphi, \tag{2.32c}$$
$$\mathring{\eth}'\,\varphi = -9\mathcal{L}_\xi\varphi{\bar{\kappa}_{1'}}^2\bar{\tau} - 3s\bar{\kappa}_{1'}\bar{\tau}\varphi + 3\bar{\kappa}_{1'}\,\eth'\,\varphi. \tag{2.32d}$$

(2) Define the vector fields

$$V^a = \frac{\Sigma}{\sqrt{2}\lambda(a^2+r^2)}l^a, \qquad Y^a = \sqrt{2}\lambda n^a. \tag{2.33}$$

Remark 2.12. The operators V and Y represent derivatives along the principal null directions, and have boost- and spin-weight zero, while the operators $\mathring{\eth}, \mathring{\eth}'$ have spin-weights $+1, -1$ respectively, but have boost-weight zero. In fact, when acting on scalars of boost- and spin-weight zero, the operators V and Y reduce to $V^a\nabla_a$ and $Y^a\nabla_a$.

The lemmas in this section all follow by direct computation.

Lemma 2.13. *The Killing vector fields ξ^a, ζ^a, and η^a, defined by (2.22), yield the following spin-weighted Lie derivative operators,*

$$\mathcal{L}_\xi\varphi = -3\kappa_1\rho'\,\text{þ}\,\varphi + 3\kappa_1\rho\,\text{þ}'\,\varphi + 3\kappa_1\tau'\,\eth\,\varphi - 3\kappa_1\tau\,\eth'\,\varphi + \tfrac{3}{2}s(\Psi_2\kappa_1 - \bar{\Psi}_2\bar{\kappa}_{1'})\varphi, \tag{2.34a}$$
$$\begin{aligned}\mathcal{L}_\zeta\varphi = {}& \tfrac{27}{4}\kappa_1(\kappa_1-\bar{\kappa}_{1'})^2(\rho'\,\text{þ}\,\varphi - \rho\,\text{þ}'\,\varphi) - \tfrac{27}{4}\kappa_1(\kappa_1+\bar{\kappa}_{1'})^2(\tau'\,\eth\,\varphi - \tau\,\eth'\,\varphi)\\ & - \tfrac{27}{8}s\big((\kappa_1+\bar{\kappa}_{1'})^2(\Psi_2\kappa_1 - \bar{\Psi}_2\bar{\kappa}_{1'}) + 8{\kappa_1}^2(-\kappa_1+\bar{\kappa}_{1'})\rho\rho'\big)\varphi,\end{aligned} \tag{2.34b}$$
$$\mathcal{L}_\eta\varphi = a^{-1}\mathcal{L}_\zeta\varphi - a\mathcal{L}_\xi\varphi. \tag{2.34c}$$

The following relation will also turn out to be useful

$$\mathcal{L}_\xi\varphi = V\varphi + \frac{\Delta}{2(a^2+r^2)}Y\varphi - \frac{a}{a^2+r^2}\mathcal{L}_\eta\varphi. \tag{2.35}$$

Definition 2.14. Define the following spin-weighted operators.

$$\widehat{R}_s = 2(a^2+r^2)YV - \frac{2ar}{a^2+r^2}\mathcal{L}_\eta + \frac{(a^4-4Ma^2r+a^2r^2+2Mr^3)}{(a^2+r^2)^2}, \tag{2.36a}$$
$$\begin{aligned}R_s = {}& 2(a^2+r^2)YV - \frac{2a(1+2s)r}{a^2+r^2}\mathcal{L}_\eta + 4srV + \frac{2Ms(a^2-r^2)}{a^2+r^2}Y\\ & + \frac{2s(M-r)r}{a^2+r^2} + \frac{\big(a^4+2Mr^3+a^2r(r-4M)\big)}{(a^2+r^2)^2},\end{aligned} \tag{2.36b}$$
$$S_s = 2(\mathring{\eth} - 9{\kappa_1}^2\tau\mathcal{L}_\xi)(\mathring{\eth}' - 9{\bar{\kappa}_{1'}}^2\bar{\tau}\mathcal{L}_\xi) - 3(2s-1)(\kappa_1-\bar{\kappa}_{1'})\mathcal{L}_\xi, \tag{2.36c}$$
$$\mathring{S}_s = 2\,\mathring{\eth}\,\mathring{\eth}', \tag{2.36d}$$
$$\widehat{\boxdot}_s = \widehat{R}_s - S_s. \tag{2.36e}$$

Remark 2.15. (1) The standard d'Alembertian is related to $\widehat{\boxdot}_s$ via

$$\nabla^a\nabla_a\varphi = \frac{1}{\Sigma\sqrt{a^2+r^2}}\widehat{\boxdot}_0(\sqrt{a^2+r^2}\varphi). \tag{2.37}$$

(2) The operator $\widehat{R}_s$ has no explicit s-dependence. In particular, $\widehat{R}_s$ coincides with the radial part of the d'Alembertian, acting on the radiation field.
(3) We have

$$[\widehat{R}_s, S_s] = [R_s, S_s] = 0. \tag{2.38}$$

(4) The operators $\widehat{R}_s, S_s$ are related to the Teukolsky radial and angular operators, cf. [47, 48, 49].
(5) Substituting $a = 0$, one finds $S_s = \mathring{S}_s$.

Lemma 2.16. *Let φ be a spin-weighted scalar.*

$$S_s\varphi = 2\mathring{\eth}\mathring{\eth}'\varphi + 2a\mathcal{L}_\eta\mathcal{L}_\xi\varphi + \tfrac{1}{4}\big(4a^2 + 9(\kappa_1 - \bar{\kappa}_{1'})^2\big)\mathcal{L}_\xi\mathcal{L}_\xi\varphi - 3s(\kappa_1 - \bar{\kappa}_{1'})\mathcal{L}_\xi\varphi. \tag{2.39a}$$

$$\overline{S_s\varphi} = S_{-s}\bar{\varphi} - 2s\bar{\varphi}. \tag{2.39b}$$

Lemma 2.17. *Let φ be a spin-weighted scalar. In the Znajek tetrad and ingoing Eddington-Finkelstein coordinates, we have*

$$V\varphi = \partial_v\varphi + \frac{\Delta\partial_r\varphi}{2(a^2+r^2)} + \frac{a\partial_\phi\varphi}{a^2+r^2}, \tag{2.40a}$$

$$Y\varphi = -\partial_r\varphi, \tag{2.40b}$$

$$\mathring{\eth}\varphi = \tfrac{1}{\sqrt{2}}\partial_\theta\varphi + \tfrac{i}{\sqrt{2}}\csc\theta\partial_\phi\varphi - \tfrac{1}{\sqrt{2}}s\cot\theta\varphi, \tag{2.40c}$$

$$\mathring{\eth}'\varphi = \tfrac{1}{\sqrt{2}}\partial_\theta\varphi - \tfrac{i}{\sqrt{2}}\csc\theta\partial_\phi\varphi + \tfrac{1}{\sqrt{2}}s\cot\theta\varphi, \tag{2.40d}$$

$$\mathcal{L}_\xi\varphi = \partial_v\varphi, \tag{2.40e}$$

$$\mathcal{L}_\eta\varphi = \partial_\phi\varphi. \tag{2.40f}$$

$$\begin{aligned} S_s\varphi = {}& a^2\sin^2\theta\partial_v\partial_v\varphi + 2a\partial_v\partial_\phi\varphi + \partial_\theta\partial_\theta\varphi + \csc^2\theta\partial_\phi\partial_\phi\varphi - 2ias\cos\theta\partial_v\varphi \\ & + \cot\theta\partial_\theta\varphi + 2is\cot\theta\csc\theta\partial_\phi\varphi + s(s - s\csc^2\theta - 1)\varphi. \end{aligned} \tag{2.40g}$$

Remark 2.18. Restricting to the sphere, spin-weighted scalars can be viewed as sections of complex line bundles. Defining spin-weighted scalars in terms of a null tetrad corresponds to a choice of local trivialization for these bundles, and the form of the operators $\mathring{\eth}, \mathring{\eth}'$ given in (2.40c) and (2.40d) are expressions, in the given trivialization and coordinate system, of covariantly defined elliptic operators of order one, acting on spin-weighted scalars on the sphere, cf. [19].

Lemma 2.19. *Let φ be a spin-weighted scalar. We have the commutator relations*

$$YV\varphi = VY\varphi + \frac{2ar}{(a^2+r^2)^2}\mathcal{L}_\eta\varphi + \frac{M(-a^2+r^2)}{(a^2+r^2)^2}Y\varphi, \qquad \mathcal{L}_\xi Y\varphi = Y\mathcal{L}_\xi\varphi, \tag{2.41}$$

and

$$Y\mathring{\eth}\varphi = \mathring{\eth}Y\varphi, \qquad Y\mathring{\eth}'\varphi = \mathring{\eth}'Y\varphi, \tag{2.42a}$$

$$V\mathring{\eth}\varphi = \mathring{\eth}V\varphi, \qquad V\mathring{\eth}'\varphi = \mathring{\eth}'V\varphi, \tag{2.42b}$$

$$\mathcal{L}_\xi\mathring{\eth}\varphi = \mathring{\eth}\mathcal{L}_\xi\varphi, \qquad \mathcal{L}_\xi\mathring{\eth}'\varphi = \mathring{\eth}'\mathcal{L}_\xi\varphi, \tag{2.42c}$$

$$\mathring{\eth}\mathring{\eth}'\varphi = \mathring{\eth}'\mathring{\eth}\varphi - s\varphi. \tag{2.42d}$$

2.4. Time functions.

Definition 2.20. We consider time functions τ defined in terms of height functions $k = k(r)$,

$$\tau = v - k(r) \tag{2.43}$$

where v is the advanced time coordinate in the ingoing Eddington-Finkelstein coordinate system.

(1) A time function τ of the form (2.43) is a regular, future hyperboloidal time function, if
  (a) $k(r)$ is smooth in an open neighbourhood of $[r_+, \infty)$.
  (b) $K(R) = k'(1/R) = k'(r)$, where $R = 1/r$, is smooth in an open neighbourhood of $[0, 1/r_+]$.
  (c) The level sets of τ are strictly spacelike in $\mathcal{M}$.

(d) The limit

$$\lim_{r\to\infty} \frac{r^2}{M^2} V^a \nabla_a \tau \tag{2.44}$$

exists and is positive.

(e)

$$\lim_{r\to\infty} Y^a \nabla_a \tau = 2. \tag{2.45}$$

(2) A time function τ of the form (2.43) with height function $k = k(r)$ is horizon crossing if
  (a) $k(r)$ is smooth in an open neighbourhood of $[r_+, \infty)$.
  (b) The level sets of τ are strictly spacelike in $\mathcal{M}$.
  (c) For large r, $k'(r) - (a^2 + r^2)/\Delta = O(r^{-2})$.

Lemma 2.21. *Let $C_{hyp} \geq 1$ and let $t = v - h(r)$ on $\mathcal{M}$, where*

$$\begin{aligned} h(r) = 2(r - r_+) + 4M \log\left(\frac{r}{r_+}\right) + \frac{3M^2(r_+ - r)^2}{r_+ r^2} + 2M \arctan\left(\frac{(C_{hyp} - 1)M}{r}\right) \\ - 2M \arctan\left(\frac{(C_{hyp} - 1)M}{r_+}\right), \end{aligned} \tag{2.46}$$

where r_+ is given by (2.26). Then t is a regular, future hyperboloidal time function as in definition 2.20. Further,

$$h(r_+) = 0, \tag{2.47a}$$

$$h'(r) \geq 0, \quad \textit{for } r \geq r_+ \tag{2.47b}$$

$$\lim_{r\to\infty} \frac{h(r)}{r} = 2, \tag{2.47c}$$

$$\lim_{r\to\infty} \frac{r^2}{M^2} V^a \nabla_a t = C_{hyp}. \tag{2.47d}$$

Proof. It is straightforward to verify (2.47a), (2.47c), and (2.47d). We have

$$h'(r) = 2 + \frac{4M}{r} + \frac{6M^2(r - r_+)}{r^3} - \frac{2(C_{\text{hyp}} - 1)M^2}{(C_{\text{hyp}} - 1)^2 M^2 + r^2}. \tag{2.48}$$

Based on this and (2.46), it is straightforward to verify points 1a, 1b of definition 2.20. Next, we prove that t has spacelike level sets. We have

$$\begin{aligned} dt_a dt_b g^{ab} \frac{\Sigma}{\Delta} &= -\frac{a^2 \sin^2\theta}{\Delta} + \frac{(a^2 + r^2)^2}{\Delta^2} - \left(h'(r) - \frac{a^2 + r^2}{\Delta}\right)^2 \\ &\geq -\frac{a^2}{\Delta} + \frac{(a^2 + r^2)^2}{\Delta^2} - \left(h'(r) - \frac{a^2 + r^2}{\Delta}\right)^2. \end{aligned} \tag{2.49}$$

Hence, t has spacelike level sets if and only if

$$0 \leq \frac{a^2 + r^2}{\Delta}\left(1 - \sqrt{1 - \frac{a^2\Delta}{(a^2 + r^2)^2}}\right) < h'(r) < \frac{a^2 + r^2}{\Delta}\left(1 + \sqrt{1 - \frac{a^2\Delta}{(a^2 + r^2)^2}}\right). \tag{2.50}$$

Using the inequality $x \leq \sqrt{x}$ for $0 \leq x \leq 1$ one finds that a sufficient (but not necessary) condition for the level sets Σ_t to be spacelike is given by

$$\frac{a^2}{a^2 + r^2} < h'(r) < \frac{2(a^2 + r^2)}{\Delta} - \frac{a^2}{a^2 + r^2}. \tag{2.51}$$

Since $C_{\text{hyp}} \geq 1$ by assumption, we have using (2.48)

$$\frac{2(a^2 + r^2)}{\Delta} - \frac{a^2}{a^2 + r^2} - h'(r) > \frac{6M^2 r_+}{r^3} + J \tag{2.52}$$

where

$$J = \frac{2(a^2 + r^2)}{\Delta} - \frac{a^2}{a^2 + r^2} - 2 - \frac{4M}{r} - \frac{6M^2}{r^2}. \tag{2.53}$$

Collecting powers of r in $\Delta(a^2+r^2)r^2J$, and using $r > r_+ > M > |a|$, one finds $J > 0$ on $\mathcal{M}$ and the right inequality in (2.51) follows. To see that the left inequality in (2.51) holds, note that

$$h'(r) - \frac{a^2}{a^2+r^2} > 2 + \frac{4M}{r} - \frac{2(C_{\text{hyp}}-1)M^2}{(C_{\text{hyp}}-1)^2M^2+r^2} - \frac{a^2}{a^2+r^2}. \tag{2.54}$$

To bound the second term of the right-hand side from below, we note that it is of the form

$$-2Mx/(x^2+r^2), \tag{2.55}$$

with $x = (C_{\text{hyp}}-1)M$. For $x > 0$, (2.55) is bounded from below by $-M/r$. Further, $a^2/(a^2+r^2) < 1$ is monotone decreasing for $r > r_+$. This gives

$$\begin{aligned} h'(r) - \frac{a^2}{a^2+r^2} &> 2 + \frac{3M}{r} - \frac{a^2}{a^2+r_+^2} \\ &\geq 1 + \frac{3M}{r} > 0. \end{aligned} \tag{2.56}$$

Hence, the level sets of t are strictly spacelike in $\mathcal{M}$. The inequality (2.56) yields (2.47b). The remaining points 1d, 1e of definition 2.20 can be verified by straightforward calculations. □

Lemma 2.22. *Let $k = h/2$ with h given by (2.46). Then $t_{\mathscr{H}^+} = v - k$ is a horizon crossing time function and*

$$k(r_+) = 0. \tag{2.57}$$

Proof. It is straightforward to verify points 2a, 2c of definition 2.20. For point 2b we proceed as in the proof of lemma 2.21, and note that a sufficient condition for $t_{\mathscr{H}^+}$ to have spacelike level sets is given by (2.51) with h' replaced by k',

$$\frac{a^2}{a^2+r^2} < k'(r) < \frac{2(a^2+r^2)}{\Delta} - \frac{a^2}{a^2+r^2}. \tag{2.58}$$

It follows from the proof of lemma 2.21 that $h' > 0$, and the second inequality in (2.58) holds since from $k = h/2$ we have that $k' < h'$. For the first inequality in (2.58), we have, following the proof of lemma 2.21,

$$\begin{aligned} k' - \frac{a^2}{a^2+r^2} &> 1 + \frac{2M}{r} - \frac{(C_{\text{hyp}}-1)M^2}{(C_{\text{hyp}}-1)^2M^2+r^2} - \frac{a^2}{a^2+r^2} \\ &> 1 + \frac{3}{2}\frac{M}{r} - \frac{a^2}{a^2+r_+^2} \\ &> 0. \end{aligned} \tag{2.59}$$

This completes the proof. □

Definition 2.23 (Time functions)**.** Define the horizon-crossing time $t_{\mathscr{H}^+}$ and the hyperboloidal time t,

$$t_{\mathscr{H}^+} = v - h/2, \tag{2.60a}$$
$$t = v - h \tag{2.60b}$$

with h as in (2.46).

Remark 2.24. (1) There is a constant c_h such that the retarded time u and the hyperboloidal time t satisfy, for large r,

$$u - t = c_h + 2C_{\text{hyp}}M^2/r + O(1/r^2). \tag{2.61}$$

(2) We have $\lim_{r\to\infty} Y^a\nabla_a u = 2$. Hence, the condition (2.45) implies that the level sets Σ_t are asymptotic to level sets of the retarded time u.

(3) There is a constant c_k such that the Boyer-Lindquist time and the horizon crossing time $t_{\mathscr{H}^+}$ satisfy, for large r,

$$t_{BL} - t_{\mathscr{H}^+} = c_k + C_{\text{hyp}}M^2/r + O(1/r^2). \tag{2.62}$$

(4) From $h(r_+) = 0$ and (2.47b) we have that $h(r) \geq 0$ for $r \geq r_+$. It follows that Σ_{t_2} is contained in the future of $\{t_{\mathscr{H}^+} = t_1\} \cap \{r > r_+\}$ precisely when $t_2 \geq t_1$.

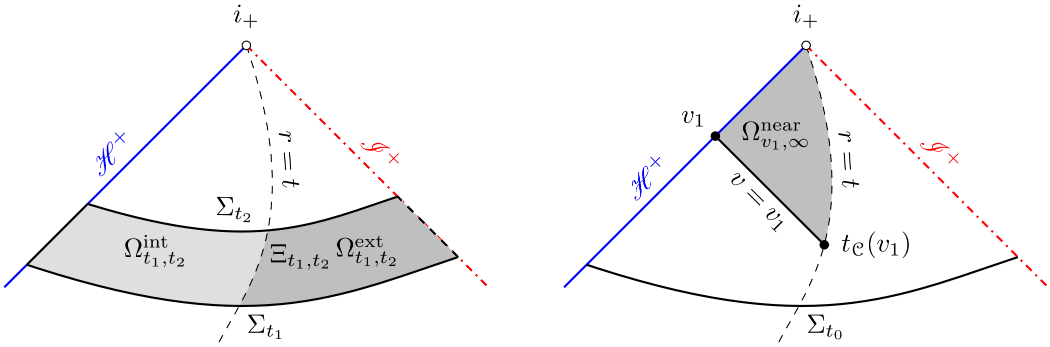

FIGURE 2. Hyperboloidal regions, cf. definition 2.26, and surfaces used for interior estimates.

(5) Although the class of hyperboloidal time functions introduced in point 1 of definition 2.20 could be employed in this paper, for simplicity we only make use of the explicit hyperboloidal time t.

Definition 2.25. (1) The future domain of dependence of a hypersurface $\Sigma \subset \mathcal{M}$ is denoted $D^+(\Sigma)$.

(2) For a subset $\Omega \in \mathcal{M}$, let $I^+(\Omega), I^-(\Omega)$ denote the time-like future and past of Ω, respectively.

Definition 2.26. (1) Define $t_0 = 10M$, and define Σ_{init} by

$$\Sigma_{\text{init}} = \{t_{\mathscr{H}^+} = t_0\} \cap \{r > r_+\}. \tag{2.63}$$

(2) Given $t_1 \in \mathbb{R}$, Σ_{t_1} denotes the corresponding level set of the hyperboloidal timefunction t, restricted to $D^+(\Sigma_{\text{init}})$,

$$\Sigma_{t_1} = \{t = t_1\} \cap D^+(\Sigma_{\text{init}}). \tag{2.64}$$

(3) Given $-\infty \leq t_1 < t_2 \leq \infty$ and $r_+ \leq r_1 < r_2$, define

$$\Sigma_t^{r_1} = \Sigma_t \cap \{r_1 \leq r\}, \tag{2.65a}$$

$$\Sigma_t^{r_1,r_2} = \Sigma_t \cap \{r_1 \leq r \leq r_2\}, \tag{2.65b}$$

$$\Omega_{t_1,t_2} = \bigcup_{t_1 \leq t \leq t_2} \Sigma_t, \tag{2.65c}$$

$$\Omega^{r_1}_{t_1,t_2} = \Omega_{t_1,t_2} \cap \{r_1 \leq r\}, \tag{2.65d}$$

$$\Omega^{r_1,r_2}_{t_1,t_2} = \Omega_{t_1,t_2} \cap \{r_1 \leq r \leq r_2\}. \tag{2.65e}$$

(4) Given $-\infty \leq t_1 < t_2 \leq \infty$, define the transition surface Ξ and a subset thereof to be

$$\Xi = \{r = t\} \cap D^+(\Sigma_{\text{init}}), \tag{2.66a}$$

$$\Xi_{t_1,t_2} = \Omega_{t_1,t_2} \cap \Xi. \tag{2.66b}$$

(5) Given $-\infty \leq t_1 < t_2 \leq \infty$, define

$$\Sigma^{\text{ext}}_{t_1} = \Sigma_{t_1} \cap \{r \geq t\}, \tag{2.67a}$$

$$\Sigma^{\text{int}}_{t_1} = \Sigma_{t_1} \cap \{r \leq t\}, \tag{2.67b}$$

$$\Omega^{\text{ext}}_{t_1,t_2} = \Omega_{t_1,t_2} \cap \{r \geq t\}, \tag{2.67c}$$

$$\Omega^{\text{int}}_{t_1,t_2} = \Omega_{t_1,t_2} \cap \{r \leq t\}. \tag{2.67d}$$

Definition 2.27. For $t \in \mathbb{R}$, define $\Omega^{\text{early}}_{\text{init},t}$ to be the intersection of the future of Σ_{init} and the past of Σ_t,

$$\Omega^{\text{early}}_{\text{init},t} = D^+(\Sigma_{\text{init}}) \cap I^-(\Sigma_t). \tag{2.68}$$

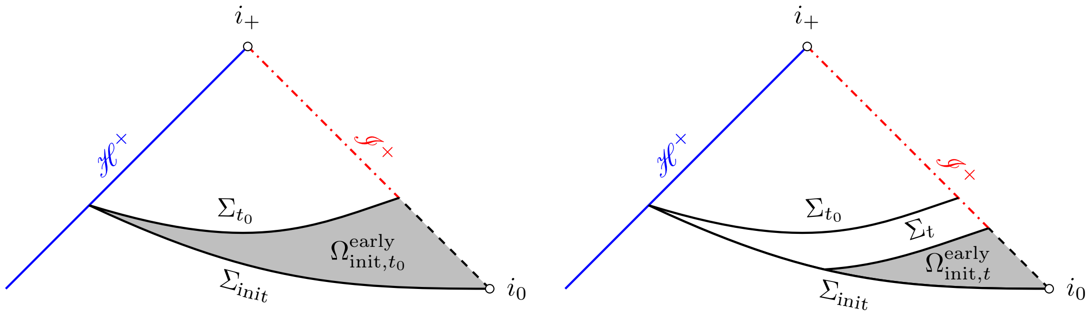

FIGURE 3. Early regions ($t < t_0$), cf. definition 2.27

Furthermore, for $r_2 > r_1 \geq r_+$, define

$$\Omega^{\text{early},r_1}_{\text{init},t} = \Omega^{\text{early}}_{\text{init},t} \cap \{r_1 \leq r\}, \tag{2.69a}$$

$$\Omega^{\text{early},r_1,r_2}_{\text{init},t} = \Omega^{\text{early}}_{\text{init},t} \cap \{r_1 \leq r \leq r_2\}. \tag{2.69b}$$

Remark 2.28. (1) For $t_1 \geq t_0$, the level set $\{t = t_1\} \cap \{r > r_+\}$ is contained in $D^+(\Sigma_{\text{init}})$, i.e. $\Sigma_{t_1} = \{t = t_1\} \cap \{r > r_+\}$.

(2) From the definition of the hyperboloidal time function, we have that on Ξ, $r + h(r) = v$. Due to (2.47b), we have that $r \mapsto r + h(r)$ defines a diffeomorphism $[t_0, \infty) \to [t_0 + h(t_0), \infty)$.

Definition 2.29. Let $v_1 \geq t_0 + h(t_0)$.

(1) Define

$$\Omega^{\text{near}}_{v_1,\infty} = \Omega^{\text{int}}_{t_0,\infty} \cap \{v \geq v_1\}, \tag{2.70}$$

where v is the advanced time.

(2) Let $t_{\mathfrak{C}}(v_1)$ be the solution to the equation

$$t_{\mathfrak{C}}(v_1) + h(t_{\mathfrak{C}}(v_1)) = v_1. \tag{2.71}$$

Remark 2.30. For v_1 as in definition 2.29, $t_{\mathfrak{C}}(v_1)$ is well defined, and the point with hyperboloidal coordinate $(t_{\mathfrak{C}}(v_1), t_{\mathfrak{C}}(v_1), \omega)$ lies on Ξ, and is the point in Ξ with ingoing Eddington-Finkelstein coordinate $(v_1, t_{\mathfrak{C}}(v_1), \omega)$. For $v_1 = t_0 + h(t_0)$, this point lies on Σ_{t_0}. Further, for $v_1 \geq t_0 + h(t_0)$, $v_1 \sim t_{\mathfrak{C}}(v_1)$.

The hypersurfaces and regions from definitions 2.26, 2.27 and 2.29 are illustrated in figures 2 and 3.

2.5. Compactified hyperboloidal coordinates.

Definition 2.31. Let (v, r, θ, ϕ) be the ingoing Eddington-Finkelstein coordinates, and let t be the hyperboloidal time function given by (2.43) with $h(r)$ given by (2.46). Define the compactified radial coordinate R to be

$$R = 1/r. \tag{2.72}$$

The compactified hyperboloidal coordinate system is (t, R, θ, ϕ). We shall write

$$H(R) = h'(r). \tag{2.73}$$

Definition 2.32. The domain of outer communication is parametrized by $(t, R, \omega) \in \mathbb{R} \times (0, r_+^{-1}) \times S^2$. For $\epsilon > 0$, this can be embedded in $\mathbb{R} \times (-\epsilon, r_+^{-1}) \times S^2$. When this is done, define, for $t_1 < t_2$,

$$\mathscr{I}^+ = \mathbb{R} \times \{0\} \times S^2, \tag{2.74a}$$

$$\mathscr{I}^+_{t_1,t_2} = [t_1, t_2] \times \{0\} \times S^2. \tag{2.74b}$$

Remark 2.33. The angular coordinates in the compactified hyperboloidal coordinate system are those of the ingoing Eddington-Finkelstein coordinates. The compactified hyperboloidal coordinates fail to be regular at $\mathscr{H}^-$.

Lemma 2.34. *In the Znajek tetrad and the compactified hyperboloidal coordinates (t, R, θ, ϕ), we have*

$$Y\varphi = H\partial_t\varphi + R^2\partial_R\varphi, \tag{2.75a}$$

$$V\varphi = \big(1 - \frac{HR^2\Delta}{2(1+a^2R^2)}\big)\partial_t\varphi - \frac{R^4\Delta\partial_R\varphi}{2(1+a^2R^2)} + \frac{aR^2\partial_\phi\varphi}{1+a^2R^2}, \tag{2.75b}$$

$$\partial_R\varphi = \frac{2a\mathcal{L}_\eta\varphi}{R^2\Delta} - \frac{2(1+a^2R^2)V\varphi}{R^4\Delta} + \frac{\mathcal{L}_\xi\varphi\big(2+2a^2R^2 - HR^2\Delta\big)}{R^4\Delta}. \tag{2.75c}$$

The operators $\mathring{\eth}, \mathring{\eth}', \mathcal{L}_\xi, \mathcal{L}_\eta, S_s$ take the form given in (2.40).

Lemma 2.35. *In the Znajek tetrad and the compactified hyperboloidal coordinate system (t, R, θ, ϕ), the operators $\widehat{R}_s$ and R_s from definition 2.14 take the form*

$$\begin{aligned}\widehat{R}_s(\varphi) = {}& \frac{H\big(2+2a^2R^2 - HR^2\Delta\big)\partial_t\partial_t\varphi}{R^2} + 2\big(1+a^2R^2 - HR^2\Delta\big)\partial_t\partial_R\varphi + 2aH\partial_t\partial_\phi\varphi \\ & - R^4\Delta\partial_R\partial_R\varphi + 2aR^2\partial_R\partial_\phi\varphi - \frac{2R\big((1+a^2R^2)^2 - MR(3+a^2R^2)\big)\partial_R\varphi}{1+a^2R^2} + \frac{2aR\partial_\phi\varphi}{1+a^2R^2} \\ & - \big(2MH(1 - \frac{2}{1+a^2R^2}) + R^2\Delta\partial_R H\big)\partial_t\varphi + \frac{R\big(a^2R + a^4R^3 + M(2-4a^2R^2)\big)\varphi}{(1+a^2R^2)^2},\end{aligned} \tag{2.76a}$$

$$\begin{aligned}R_s(\varphi) = {}& \frac{H\big(2+2a^2R^2 - HR^2\Delta\big)\partial_t\partial_t\varphi}{R^2} + 2\big(1+a^2R^2 - HR^2\Delta\big)\partial_t\partial_R\varphi + 2aH\partial_t\partial_\phi\varphi \\ & - R^4\Delta\partial_R\partial_R\varphi + 2aR^2\partial_R\partial_\phi\varphi - 2R\Big(1 + s(1-MR) - R\Big(M - a^2R + \frac{2M}{1+a^2R^2}\Big)\Big)\partial_R\varphi \\ & + \frac{\Big(2H\big(MR - Ma^2R^3 - s(1-MR)(1+a^2R^2)\big) + (1+a^2R^2)\big(4s - R^3\Delta H'\big)\Big)\partial_t\varphi}{R + a^2R^3} \\ & + \frac{2aR\partial_\phi\varphi}{1+a^2R^2} + \frac{\big(2MR - 2s(1-MR)(1+a^2R^2) + a^2R^2(1-4MR+a^2R^2)\big)\varphi}{(1+a^2R^2)^2}.\end{aligned} \tag{2.76b}$$

3. The linearized Einstein equation

3.1. **First-order form of the linearized Einstein equations.** Let δg_{ab} be a solution of the linearized Einstein equations on $(\mathcal{M}, g_{ab})$, and let $G_{ABA'B'}$ and $\not{G}$ be its trace-free and trace parts respectively. The trace-free part has components[6]

$$G_{00'} = G_{ab}l^a l^b, \quad G_{10'} = G_{ab}l^a\bar{m}^b, \quad G_{11'} = G_{ab}l^a n^b, \tag{3.1a}$$

$$G_{20'} = G_{ab}\bar{m}^a\bar{m}^b, \quad G_{21'} = G_{ab}n^a\bar{m}^b, \quad G_{22'} = G_{ab}n^a n^b \tag{3.1b}$$

and their complex conjugates. We have that $G_{00'}, G_{11'}, G_{22'}$ are real, while the remaining components are complex.

The linearized connection is given by covariant derivatives of $G_{ABA'B'}$ and $\not{G}$ and has irreducible parts

$$\not{\phi}_{CA'} = \tfrac{1}{4}\nabla^{AB'}G_{CAA'B'} - \tfrac{3}{16}\nabla_{CA'}\not{G}, \qquad \text{♀}_{ABCA'} = -\tfrac{1}{2}\nabla_{(A}{}^{B'}G_{BC)A'B'}. \tag{3.2}$$

We now formulate, following [11], the linearized versions of a commutator relation, the vacuum Ricci relations, and the vacuum Bianchi identity. The quantity $\text{♀}_{ABCA'}$ introduced in (3.2) is

[6] Recall that $\varphi_{ii'}$ denotes the dyad component of a symmetric spinor $\varphi_{AB\cdots DA'B'\cdots D'}$ defined by contracting i times with ι^A and i' times with $\iota^{A'}$ as explained in section 2.1.

the symmetrized part of the spinor $\Gamma_{AA'BC}$ used there. Let $\vartheta\Psi_{ABCD}$ be the covariant linearized Weyl spinor in the sense of [11]. The relations and identities are

$$\nabla^{CA'}\Gamma_{ABCA'} = -\tfrac{4}{3}\nabla_{(A}{}^{A'}\slashed{\phi}_{B)A'}, \tag{3.3a}$$

$$\nabla^{AA'}\slashed{\phi}_{AA'} = 0, \tag{3.3b}$$

$$\nabla^{C(A'}\Gamma_{ABC}{}^{B')} = \tfrac{1}{2}G^{CDA'B'}\Psi_{ABCD} + \tfrac{2}{3}\nabla_{(A}{}^{(A'}\slashed{\phi}_{B)}{}^{B')}, \tag{3.3c}$$

$$\nabla_{(A}{}^{A'}\Gamma_{BCD)A'} = -\tfrac{1}{4}\Psi_{ABCD}\slashed{G} - \vartheta\Psi_{ABCD}, \tag{3.3d}$$

$$\nabla^{D}{}_{A'}\vartheta\Psi_{ABCD} = 2\Psi_{ABCD}\slashed{\phi}^{D}{}_{A'} + \tfrac{1}{2}(\nabla_{FB'}\Psi_{ABCD})G^{DF}{}_{A'}{}^{B'} + 3\Psi_{(AB}{}^{DF}\Gamma_{C)DFA'}. \tag{3.3e}$$

Define the following linear combinations of the components of the linearized connection,

$$\tilde{\beta} = -\tfrac{1}{3}\slashed{\phi}_{01'} + \Gamma_{11'}, \quad \tilde{\beta}' = -\tfrac{1}{3}\slashed{\phi}_{10'} - \Gamma_{20'}, \quad \tilde{\epsilon} = -\tfrac{1}{3}\slashed{\phi}_{00'} + \Gamma_{10'}, \quad \tilde{\epsilon}' = -\tfrac{1}{3}\slashed{\phi}_{11'} - \Gamma_{21'}, \tag{3.4a}$$

$$\tilde{\kappa} = \Gamma_{00'}, \quad \tilde{\kappa}' = -\Gamma_{31'}, \quad \tilde{\rho} = \tfrac{2}{3}\slashed{\phi}_{00'} + \Gamma_{10'}, \quad \tilde{\rho}' = \tfrac{2}{3}\slashed{\phi}_{11'} - \Gamma_{21'}, \tag{3.4b}$$

$$\tilde{\sigma} = \Gamma_{01'}, \quad \tilde{\sigma}' = -\Gamma_{30'}, \quad \tilde{\tau} = \tfrac{2}{3}\slashed{\phi}_{01'} + \Gamma_{11'}, \quad \tilde{\tau}' = \tfrac{2}{3}\slashed{\phi}_{10'} - \Gamma_{20'}. \tag{3.4c}$$

The notation used here is inspired by the notation introduced for spin coefficients by Geroch, Held and Penrose [24]. Note that the scalars defined in (3.4) are only the leading order terms of the linearized spin coefficients, but as they are just components of the spinors $\slashed{\phi}_{AA'}$ and $\Gamma_{ABCA'}$, they have proper GHP weights in contrast to the linearized spin coefficients. Also note that none of the quantities we study here depends on the linearized frame rotations. See appendix A for the first order system of scalar equations which results.

3.2. **Outgoing radiation gauge.**

Definition 3.1. Let δg_{ab} be a linearized metric on $(\mathcal{M}, g_{ab})$. We say that δg_{ab} satisfies the $\delta g \cdot n$ condition if

$$\delta g_{ab} n^b = 0, \tag{3.5}$$

and the trace-free condition if

$$g^{ab}\delta g_{ab} = 0. \tag{3.6}$$

If both (3.5) and (3.6) hold, then δg_{ab} is said to be in outgoing radiation gauge (ORG). Replacing n^a by l^a yields the ingoing radiation gauge (IRG) condition.

Lemma 3.2 (Price, Shankar and Whiting [40])**.** *Let δg_{ab} be a solution of the linearized vacuum Einstein equation on $(\mathcal{M}, g_{ab})$. There is a vector field ν^a such that the gauge transformed metric*

$$\delta g_{ab} - 2\nabla_{(a}\nu_{b)} \tag{3.7}$$

is in ORG.

Remark 3.3.
1. The $\delta g \cdot n$ gauge condition (3.5), which consist of four conditions, can be imposed for a general linearized metric on a background vacuum spacetime with repeated principal null direction n^a, by sequentially solving a system of four scalar equations, cf. [40, Eq. (15)]. The analogous statement is valid for the $\delta g \cdot l$ condition. This is in contrast to the ORG or IRG conditions, which contain five conditions, and which can be imposed only for linearized metrics on algebraically special background spacetimes, provided that the linearized Einstein tensor satisfies additional conditions. In [40], it is shown to be possible to impose IRG for solutions of the linearized Einstein equations $\delta E_{ab} = 8\pi\delta T_{ab}$ on a Petrov type II or type D background with repeated principal vector l^a, provided $\delta T_{ab} l^a l^b = 0$. Analogously the ORG condition can be imposed provided $\delta T_{ab} n^a n^b = 0$. Here we shall be interested only in the case of solutions of the linearized vacuum Einstein equations $\delta E_{ab} = 0$ on the Kerr spacetime, which is Petrov type D.
2. Imposing the gauge condition does not determine the vector field ν^a uniquely. In particular, there remains residual gauge degrees of freedom in ν^a, subject to constraint equations. The vector field ν^a can determined uniquely along the flow lines of n^a by specifying its initial values on a hypersurface.
3. The gauge vector field ν^a plays no explicit role in this paper.

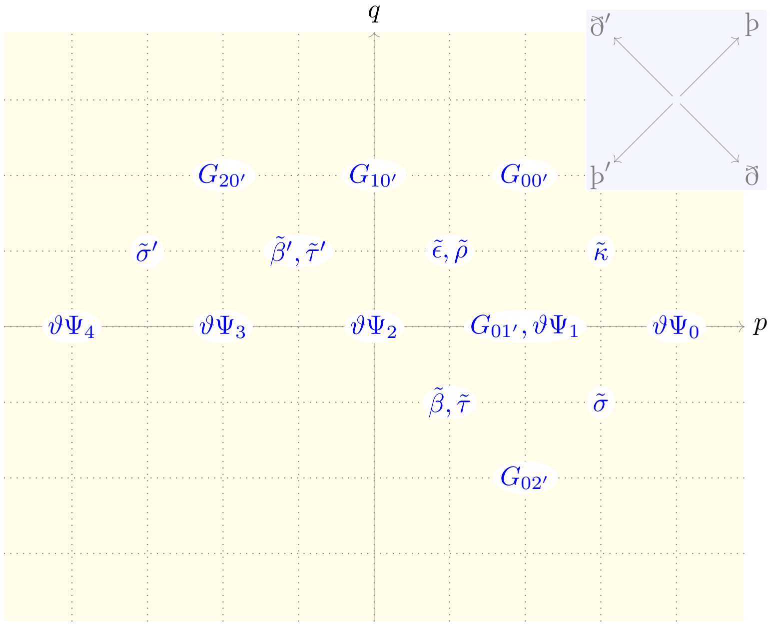

FIGURE 4. GHP weights of the nonvanishing components in ORG.

Lemma 3.4. *Let δg_{ab} be a solution to the vacuum linearized Einstein equations on $(\mathcal{M}, g_{ab})$, in ORG. Then, in the notation introduced in section 3.1, the following holds.*

(1)

$$\not{G} = 0, \qquad G_{11'} = 0, \qquad G_{12'} = 0, \tag{3.8a}$$
$$G_{21'} = 0, \qquad G_{22'} = 0, \tag{3.8b}$$

and

$$\tilde{\epsilon}' = 0, \qquad \tilde{\kappa}' = 0, \qquad \tilde{\rho}' = 0. \tag{3.9}$$

(2) The only nonvanishing components of the metric are

$$G_{00'} = \delta g_{ab} l^a l^b, \quad G_{10'} = \delta g_{ab} l^a \bar{m}^b, \quad G_{20'} = \delta g_{ab} \bar{m}^a \bar{m}^b. \tag{3.10}$$

Proof. These follow by direct computation. □

The nonvanishing linearized metric, connection, and curvature components in the outgoing radiation gauge are illustrated with their $\{p, q\}$ type in figure 4.

3.3. **Equations of linearized gravity in the boost-weight zero formalism.** From the system in appendix A, one can derive a system of transport equations relating the metric components to the Teukolsky variable $\vartheta\Psi_4$ via certain components of the linearized connection.

In order to perform estimates, it is useful to work with spin-weighted scalars, i.e. properly weighted scalars with boost weight zero, since the modulus of a spin-weighted scalar is a true scalar with boost and spin weight zero. Motivated by the above discussion, we shall derive a transport system for spin-weighted quantities. The system is written in terms of the spin-weighted operator Y introduced in section 2.3.

Definition 3.5. Let δg_{ab} be a solution to the linearized vacuum Einstein equation on the Kerr exterior $(\mathcal{M}, g_{ab})$ and let $\vartheta\Psi_0, \vartheta\Psi_4$ be the components of the linearized Weyl spinor $\vartheta\Psi_{ABCD}$ of boost- and spin-weights $(2, 2), (-2, -2)$. Define

$$\hat{\psi}_{-2} = \tfrac{1}{2}\sqrt{a^2 + r^2}\,\lambda^2 \vartheta\Psi_4, \tag{3.11a}$$
$$\hat{\psi}_{+2} = \tfrac{1}{2}\sqrt{a^2 + r^2}\,(3\kappa_1)^4 \lambda^{-2} \vartheta\Psi_0, \tag{3.11b}$$

where λ is given by definition 2.9.

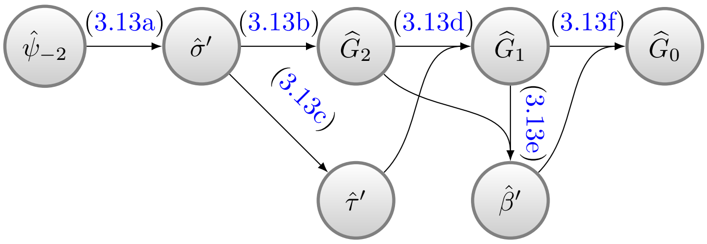

FIGURE 5. Structure of transport equations.

Remark 3.6. The Weyl scalars $\vartheta\Psi_0, \vartheta\Psi_4$ are given in terms of the linearized Weyl tensor by equation (1.12). The fields $\hat{\psi}_{-2}$ and $\hat{\psi}_{+2}$ have boost-weight zero and spin-weights -2 and $+2$, respectively.

Definition 3.7. Define the spin-weighted scalars

$$\hat{\sigma}' = \frac{\tilde{\sigma}'}{\bar{\rho}'}, \qquad \widehat{G}_2 = G_{20'}\overline{\kappa}_{1'}, \tag{3.12a}$$

$$\hat{\tau}' = \big(1 + \frac{\kappa_1}{2\overline{\kappa}_{1'}}\big)\tilde{\tau}' - \tilde{\beta}', \qquad \widehat{G}_1 = \frac{G_{10'}\kappa_1{}^3\overline{\kappa}_{1'}\rho'}{r}, \tag{3.12b}$$

$$\hat{\beta}' = \overline{\kappa}_{1'}(\tilde{\beta}' - \tfrac{1}{2}G_{10'}\bar{\rho}' + \tfrac{1}{2}G_{20'}\bar{\tau}' - \tilde{\tau}'), \qquad \widehat{G}_0 = \frac{G_{00'}\kappa_1{}^3\overline{\kappa}_{1'}\rho'^2}{r}. \tag{3.12c}$$

Lemma 3.8. *Given a solution to the linearized vacuum Einstein equation in ORG on* $(\mathcal{M}, g_{ab})$*, let the quantities* $\hat{\sigma}', \widehat{G}_2, \hat{\tau}', \widehat{G}_1, \hat{\beta}', \widehat{G}_0$ *be as in definition 3.7, and let* $\hat{\psi}_{-2}$ *be as in definition 3.5. Then we have*

$$Y(\hat{\sigma}') = -\frac{12\overline{\kappa}_{1'}\hat{\psi}_{-2}}{\sqrt{r^2+a^2}}, \tag{3.13a}$$

$$Y(\widehat{G}_2) = -\tfrac{2}{3}\hat{\sigma}', \tag{3.13b}$$

$$Y(\hat{\tau}') = -\frac{\kappa_1(\eth - 2\tau + 2\bar{\tau}')\hat{\sigma}'}{6\overline{\kappa}_{1'}{}^2}, \tag{3.13c}$$

$$Y(\widehat{G}_1) = \frac{2\kappa_1{}^2\overline{\kappa}_{1'}{}^2\hat{\tau}'}{r^2} + \frac{\kappa_1{}^2\overline{\kappa}_{1'}(\eth - \tau + \bar{\tau}')\widehat{G}_2}{2r^2}, \tag{3.13d}$$

$$Y(\hat{\beta}') = \frac{r\widehat{G}_1}{6\kappa_1{}^2\overline{\kappa}_{1'}{}^2} + \frac{\kappa_1\tau\widehat{G}_2}{6\overline{\kappa}_{1'}{}^2}, \tag{3.13e}$$

$$Y(\widehat{G}_0) = -\frac{(\eth - \tau)\widehat{G}_1}{3\kappa_1} - \frac{\tau\widehat{G}_1}{r} - \frac{\bar{\tau}\overline{\widehat{G}_1}}{r} + \frac{2\kappa_1{}^2\overline{\kappa}_{1'}(\eth - \bar{\tau}')\hat{\beta}'}{r^2} - \frac{(\eth' - \bar{\tau})\overline{\widehat{G}_1}}{3\overline{\kappa}_{1'}} + \frac{2\kappa_1\overline{\kappa}_{1'}{}^2(\eth' - \tau')\overline{\hat{\beta}'}}{r^2}. \tag{3.13f}$$

Remark 3.9.
(1) The quantities $\hat{\sigma}', \widehat{G}_2, \hat{\tau}', \widehat{G}_1, \hat{\beta}', \widehat{G}_0$ have spin-weights $-2, -2, -1, -1, -1, 0$, respectively.
(2) The definition of the quantities $\widehat{G}_0$ and $\widehat{G}_1$ has the consequence that the linearized mass δM and angular momentum per unit mass δa appear as constants of integration in equations (3.13f) and (3.13d), respectively. In section 8.4 we show that our assumptions imply that these constant vanish. The choice of $\hat{\tau}'$ is made so that it vanishes for a linearized mass or angular momentum perturbation in ORG. See appendix B.

Proof. Throughout the proof we will use the relations

$$\text{þ}'\kappa_1 = -\kappa_1\rho', \qquad \text{þ}'\overline{\kappa}_{1'} = -\overline{\kappa}_{1'}\bar{\rho}', \qquad \text{þ}'\rho' = \rho'^2, \qquad \text{þ}'\bar{\rho}' = \bar{\rho}'^2, \tag{3.14a}$$

$$\eth\kappa_1 = -\kappa_1\tau, \qquad \eth\overline{\kappa}_{1'} = \kappa_1\tau, \qquad \eth\rho' = 2\rho'\tau, \qquad \eth\bar{\rho}' = \bar{\rho}'\tau + \bar{\rho}'\bar{\tau}', \tag{3.14b}$$

$$\overline{\kappa}_{1'}\bar{\rho} = \kappa_1\rho, \qquad \overline{\kappa}_{1'}\bar{\rho}' = \kappa_1\rho', \qquad \overline{\kappa}_{1'}\bar{\tau}' = -\kappa_1\tau, \qquad \overline{\kappa}_{1'}\bar{\tau} = -\kappa_1\tau'. \tag{3.14c}$$

For some calculations it might also be worth to notice

$$\eth\,\tau=\tau^2,\quad \eth'\,\tau=\tfrac{1}{2}\Psi_2-\frac{\bar\Psi_2\bar\kappa_{1'}}{2\kappa_1}+\rho\rho'-\frac{\kappa_1\rho\rho'}{\bar\kappa_{1'}}+\tau\tau'=\frac{(a^2+r^2)(\kappa_1-\bar\kappa_{1'})}{162\kappa_1{}^3\bar\kappa_{1'}{}^2}+\tau\tau', \tag{3.15a}$$

$$\eth'\,\tau'=\tau'^2,\quad \eth\,\tau'=\tfrac{1}{2}\Psi_2-\frac{\bar\Psi_2\bar\kappa_{1'}}{2\kappa_1}+\rho\rho'-\frac{\kappa_1\rho\rho'}{\bar\kappa_{1'}}+\tau\tau'=\frac{(a^2+r^2)(\kappa_1-\bar\kappa_{1'})}{162\kappa_1{}^3\bar\kappa_{1'}{}^2}+\tau\tau'. \tag{3.15b}$$

The ORG condition reduces equations (A.3e) and (A.1j) to the transport equations

$$(\text{þ}'-\bar\rho')\tilde\sigma'=\vartheta\Psi_4,\qquad\qquad (\text{þ}'-\bar\rho')G_{20'}=2\tilde\sigma'. \tag{3.16}$$

The choices made in definition 3.7 are explained below. First we note that $\tilde\sigma'$ has boost weight, which is eliminated by defining $\hat\sigma'=\frac{\tilde\sigma'}{\bar\rho'}$. Similarly the definition $\widehat{G}_2=\overline{\kappa}_{1'}G_{20'}$ compensates for the lower-order term in the left-hand side. We then re-express the transport equations in terms of the spin-weighted operator Y defined in (2.32b), from which we get (3.13a) and (3.13b).

Under the ORG conditions, the equations (A.2j) and (A.4c) will yield expressions for $\text{þ}'\,\tilde\tau'$ and $\text{þ}'\,\tilde\beta'$. However, these transport equations are coupled, so we need to change variables. The corresponding transport equation then reduces to

$$\text{þ}'\,\hat\tau'=\frac{\kappa_1(\eth-3\tau+\bar\tau')\tilde\sigma'}{2\overline{\kappa}_{1'}}, \tag{3.17}$$

which can be written as (3.13c). Using equations (A.1b) and (A.1l) to express $\hat\tau'$ in terms of $G_{10'}$ and $G_{20'}$ yields

$$\hat\tau'=-\frac{r\,\text{þ}'\,G_{10'}}{6\overline{\kappa}_{1'}}-\frac{(\kappa_1{}^2+\kappa_1\overline{\kappa}_{1'}+2\overline{\kappa}_{1'}{}^2)\rho' G_{10'}}{4\overline{\kappa}_{1'}{}^2}-\tfrac{1}{4}(\eth-\tau)G_{20'}, \tag{3.18}$$

which can be rewritten as a transport equation for $G_{10'}$. Furthermore, one can rescale $G_{10'}$ to produce a boost-weight zero quantity $\widehat{G}_1$ such that the contribution from the linearized angular momentum in ORG gauge is r independent, cf. section B.1. This also eliminates the lower-order terms to yield the transport equation (3.13d).

The transport equation for $\tilde\beta'$ is complicated. $\hat\beta'=\bar\kappa_{1'}\overline{\tilde\beta}$ satisfies a much simpler equation arising from the complex conjugate of (A.2c) subject to the ORG conditions (3.8) and (3.9). The rescaling eliminates the lower-order term. However, $\hat\beta'$ can be reexpressed in terms of $\tilde\beta'$, and the already controlled quantities $\hat\tau'$, $\hat G_1$, and $\hat G_0$ using (A.5) and

$$\tilde\tau=-\tfrac{1}{2}\rho' G_{01'}+\tfrac{1}{2}\tau' G_{02'}, \tag{3.19}$$

which follows from (A.1k) and the ORG conditions.

Taking a derivative of equation (3.19) and using the relations (A.1), (A.5) and the definitions of $\hat\tau'$ and $\hat\beta'$ yield

$$\eth'\,\tilde\tau=\tfrac{1}{2}\rho'\bar\rho' G_{00'}+\rho'\overline{\tilde\rho}+\tfrac{1}{2}\bar\rho'\tau G_{10'}+\frac{4r\bar\tau\overline{\hat\beta'}}{3\kappa_1{}^2}+\tfrac{1}{2}\bar\rho'\tau' G_{01'}+\bar\tau\tau' G_{02'}-2\bar\tau\overline{\tilde\tau'}. \tag{3.20}$$

The relations (A.2a), (A.2h) and (A.4b) together give

$$0=-\rho'\tilde\epsilon-\bar\rho'\tilde\epsilon+\rho'\tilde\rho+2\tau'\tilde\beta-\bar\tau'\tilde\tau'-\eth\,\tilde\beta'+\eth\,\tilde\tau'-\eth'\,\tilde\beta. \tag{3.21}$$

This together with the definitions of $\hat\tau'$ and $\hat\beta'$ and (3.20) yields

$$0=-\rho'\tilde\epsilon-\bar\rho'\tilde\epsilon+\tfrac{1}{2}\rho'\bar\rho' G_{00'}+\rho'\tilde\rho+\bar\rho'\tilde\rho-\frac{\kappa_1\tau\hat\beta'}{\overline{\kappa}_{1'}{}^2}+\tfrac{1}{2}\bar\rho'\tau G_{10'}+\frac{\tau'\overline{\hat\beta'}}{\kappa_1}-\tfrac{1}{2}\bar\rho'\tau' G_{01'}-\frac{\eth\,\hat\beta'}{\overline{\kappa}_{1'}}-\frac{\eth'\,\overline{\hat\beta'}}{\kappa_1}. \tag{3.22}$$

With the help of equations (A.1c) and (A.1g), this can be rewritten as a transport equation for $G_{00'}$

$$\begin{aligned}\text{þ}'\,G_{00'}={}&-\frac{3(\kappa_1{}^2+\overline{\kappa}_{1'}{}^2)\rho' G_{00'}}{2r\overline{\kappa}_{1'}}-\frac{3(\kappa_1{}^2-3\kappa_1\overline{\kappa}_{1'}-2\overline{\kappa}_{1'}{}^2)\tau G_{10'}}{2r\overline{\kappa}_{1'}}-\frac{6\kappa_1\tau\hat\beta'}{r\overline{\kappa}_{1'}\rho'}\\&-\frac{3(2\kappa_1{}^2+3\kappa_1\overline{\kappa}_{1'}-\overline{\kappa}_{1'}{}^2)\tau' G_{01'}}{2r\overline{\kappa}_{1'}}+\frac{6\overline{\kappa}_{1'}\tau'\overline{\hat\beta'}}{r\kappa_1\rho'}+\eth\,G_{10'}-\frac{6\,\eth\,\hat\beta'}{r\rho'}+\eth'\,G_{01'}-\frac{6\overline{\kappa}_{1'}\,\eth'\,\overline{\hat\beta'}}{r\kappa_1\rho'}.\end{aligned} \tag{3.23}$$

This can then be expressed as (3.13f) in terms of spin-weighted quantities, where the scaling of $\widehat{G}_0$ was chosen such that the contribution of the linearized mass in ORG gauge is constant, cf. (B.3). This also eliminates the lower-order terms to yield the transport equation (3.13f). ☐

3.4. **The Teukolsky equations.** In GHP form, the Teukolsky Master Equations take the form

$$\big((\text{þ} - 3\rho - \bar{\rho})\,\text{þ}' - (\eth - 3\tau - \bar{\tau}')\,\eth' - 3\Psi_2\big)(\kappa_1\vartheta\Psi_0) = 0, \tag{3.24a}$$

$$\big((\text{þ}' - 3\rho' - \bar{\rho}')\,\text{þ} - (\eth' - 3\tau' - \bar{\tau})\,\eth - 3\Psi_2\big)(\kappa_1\vartheta\Psi_4) = 0, \tag{3.24b}$$

in the source-free case, cf. [2, Eqs. (A.2)]. A calculation shows that in terms of the de-boosted variables $\hat{\psi}_{\pm 2}$, equations (3.24) take the form given in the following lemma.

Lemma 3.10. *Let $\hat{\psi}_{-2}, \hat{\psi}_{+2}$ be as in definition 3.5.*

$$\widehat{\boxed{\text{S}}}_{-2}(\hat{\psi}_{-2}) = -\frac{8ar}{a^2+r^2}\mathcal{L}_\eta\hat{\psi}_{-2} + 8rV\hat{\psi}_{-2} + \frac{4M(a^2-r^2)}{a^2+r^2}Y\hat{\psi}_{-2} + \frac{4(M-r)r\hat{\psi}_{-2}}{a^2+r^2}, \tag{3.25a}$$

$$\widehat{\boxed{\text{S}}}_{2}(\hat{\psi}_{+2}) = \frac{8ar}{a^2+r^2}\mathcal{L}_\eta\hat{\psi}_{+2} - 8rV\hat{\psi}_{+2} - \frac{4M(a^2-r^2)}{a^2+r^2}Y\hat{\psi}_{+2} + \frac{4r(r-M)\hat{\psi}_{+2}}{a^2+r^2}. \tag{3.25b}$$

3.5. **The Teukolsky-Starobinsky Identities.** The two main Teukolsky-Starobinsky Identities for linearized gravity [2, Eqs. (A.5a), (A.5e)] are

$$0 = \text{þ}'\,\text{þ}'\,\text{þ}'\,\text{þ}'(\kappa_1^4\vartheta\Psi_0) - \eth\,\eth\,\eth\,\eth(\kappa_1^4\vartheta\Psi_4) - \tfrac{M}{27}\mathcal{L}_\xi\overline{\vartheta\Psi_4}, \tag{3.26a}$$

$$0 = \eth'\,\eth'\,\eth'\,\eth'(\kappa_1^4\vartheta\Psi_0) - \text{þ}\,\text{þ}\,\text{þ}\,\text{þ}(\kappa_1^4\vartheta\Psi_4) - \tfrac{M}{27}\mathcal{L}_\xi\overline{\vartheta\Psi_0}. \tag{3.26b}$$

See [2] for the complete set of 5 Teukolsky-Starobinsky Identities for linearized gravity on Petrov type D spacetimes. Define the spin-weight 1 quantity $\mathring{\tau}$ by

$$\mathring{\tau} = -9{\kappa_1}^2\tau, \tag{3.27}$$

where τ is the GHP spin-coefficient. Then $\mathring{\tau}$ satisfies

$$\mathring{\eth}(\mathring{\tau}) = 0, \qquad \mathcal{L}_\xi(\mathring{\tau}) = 0, \qquad Y(\mathring{\tau}) = 0. \tag{3.28}$$

The following lemma is proved by a calculation, starting from equation (3.26a).

Lemma 3.11. *In terms of the variables $\hat{\psi}_{-2}$ and $\hat{\psi}_{+2}$ introduced in definition 3.5 and the spin-weighted operators introduced in definition 2.11, we have*

$$\mathring{\eth}^4\,\hat{\psi}_{-2} = -3M\mathcal{L}_\xi(\overline{\hat{\psi}_{-2}}) - \sum_{k=1}^{4}\binom{4}{k}\mathring{\tau}^k\,\mathring{\eth}^{4-k}\,{\mathcal{L}_\xi}^k\hat{\psi}_{-2} + \tfrac{1}{4}\Big(Y + \frac{r}{a^2+r^2}\Big)^4\hat{\psi}_{+2}. \tag{3.29}$$

4. Analytic preliminaries

4.1. **Conventions and notation.** The set of natural numbers $\{0, 1, \dots\}$ is denoted $\mathbb{N}$, the integers $\mathbb{Z}$, and the positive integers $\mathbb{Z}^+$. Recall that $t_0 = 10M$ was set in definition 2.27.

Definition 4.1. Throughout the rest of the paper, let $C_{\text{hyp}} = 10^6$ be fixed[7].

Definition 4.2. The reference volume forms are

$$\text{d}^2\mu = \sin\theta\text{d}\theta \wedge \text{d}\phi, \tag{4.1a}$$

$$\text{d}^3\mu = \text{d}r \wedge \text{d}^2\mu, \tag{4.1b}$$

$$\text{d}^4\mu = \text{d}t \wedge \text{d}^3\mu, \tag{4.1c}$$

$$\text{d}^3\mu_{\mathscr{I}} = \text{d}t \wedge \text{d}^2\mu. \tag{4.1d}$$

Given a 1-form ν, let $\text{d}^3\mu_\nu$ denote a Leray 3-form such that $\nu \wedge \text{d}^3\mu_\nu = \text{d}^4\mu$, see [22].

[7]The results of this paper holds for any sufficiently large C_{hyp}.

Remark 4.3. The family of Kerr metrics, when written for example in ingoing Eddington-Finkelstein coordinates, are such that, for any $\Lambda > 0$, the rescaling

$$(M, a, v, r, \theta, \phi) \mapsto (\Lambda M, \Lambda a, \Lambda v, \Lambda r, \theta, \phi) \tag{4.2}$$

takes a Kerr solution to a Kerr solution. Thus, if an estimate can be proved for a given value of $M = M_1$, then the same estimate can be proved for another value $M = M_2$ by rescaling with $\Lambda = M_2/M_1$. Furthermore, any statement in this paper involving (a, v, r) can be restated for any given M as a statement in terms of $(a/M, v/M, r/M)$. It follows from the definition of the hyperboloidal time function that it scales as

$$t \to \Lambda t \tag{4.3}$$

with respect to the rescaling (4.2).

Definition 4.4. (1) We say that a quantity Q has dimension M^μ if $Q \to \Lambda^\mu Q$ under a rescaling of the type (4.2). In particular, Q is said to be dimensionless if $\mu = 0$.
(2) In view of remark 4.3 it is sufficient to consider $M = 1$. This procedure will be referred to as mass normalization.

Definition 4.5. (1) Let δ denote a sufficiently small, positive constant.
(2) We shall use regularity parameters, generally denoted k, and sufficiently large regularity constants K, independent of $k, |a|/M, \delta$.
(3) Unless otherwise specified, we shall in estimates use constants $C = C(k, |a|/M, \delta)$.
(4) Let P be a set of parameters. A constant $C(P)$ is a constant of the form

$$C(P) = C(P; k, |a|/M, \delta). \tag{4.4}$$

Remark 4.6. (1) Throughout this paper, it is necessary to have many small parameters. It is sufficient to replace all of these small parameters by the smallest of them and, hence, to treat them all as a single parameter. This small parameter is denoted $\delta > 0$.
(2) Unless otherwise stated, constants such as C, K can change value from line to line, as needed, and the allowed range of values for δ may decrease as needed.

Definition 4.7. (1) Let F_1, F_2 be dimensionless quantities, and let δ be a positive dimensionless constant. We say that $F_1 \lesssim F_2$ if there exists a constant C such that $F_1 \le CF_2$.
(2) Let F_1, F_2 be such that F_1/F_2 has dimension M^γ. We say that $F_1 \lesssim F_2$ if $F_1 \le M^\gamma C F_2$.
(3) Let P be a set of parameters. We say that $F_1 \lesssim_P F_2$ if there is a constant $C(P)$ such that $F_1 \lesssim C(P)F_2$.
(4) We say that $F_1 \gtrsim F_2$ and $F_1 \gtrsim_P F_2$ if $F_2 \lesssim F_1$ and $F_2 \lesssim_P F_1$, respectively, and further that $F_1 \sim F_2$ if it holds that $F_1 \lesssim F_2$ and $F_2 \lesssim F_1$. For a set of parameters P, $F_1 \sim_P F_2$ is defined analogously.

Definition 4.8. Let $m \in \mathbb{N}$.
(1) Let R be the compactified radial coordinate. We say that $f(R, \omega) = O_\infty(R^m)$ if $\forall j \in \mathbb{N}$,

$$|\partial_R^j f(R)| \le C(j) R^{\max\{m-j,0\}} \quad \text{for } R \in (0, 1/10M]. \tag{4.5}$$

(2) We say that $f(r, \omega) = O_\infty(r^{-m})$ if $f(R) = O_\infty(R^m)$.

Definition 4.9. For any $\gamma \in \mathbb{R}$, a bound involving the expression $\gamma-$ means that there is a constant $C > 0$, not depending on $k, |a|/M, \delta$, such that the bound holds with $\gamma-$ replaced by $\gamma - C\delta$. Similarly, a bound involving the expression $\gamma+$ means that there is a constant $C > 0$ such that the bound holds with $\gamma+$ replaced by $\gamma + C\delta$.

Definition 4.10. Let t be the hyperboloidal time function from definition 2.23. Define

$$\langle t \rangle = (M^2 + t^2)^{1/2}. \tag{4.6}$$

4.2. Conformal regularity.

Definition 4.11. A spin-weighted scalar φ is said to be conformally regular if it is smooth in the future domain of dependence of Σ_{init} and extends smoothly to $\mathbb{R} \times [-\epsilon, r_+^{-1}) \times S^2$ in the compactified hyperboloidal coordinates (t, R, ω), for some $\epsilon > 0$. A differential operator is conformally regular if it has an extension that maps conformally regular scalars to conformally regular scalars.

Lemma 4.12. *The coefficient H from definition 2.31 which arises in considering Σ_t satisfies*

$$(2+2a^2R^2-H)R^2\Delta = 2C_{hyp}M^2R^2 + M^3O_\infty(R^3). \tag{4.7}$$

In the Znajek tetrad and the compactified hyperboloidal coordinate system (t,R,θ,ϕ), we have for a spin-weighted scalar φ, which is smooth at $R=0$,

$$\partial_R\varphi = -2R^{-2}V\varphi + MR^{-1}O_\infty(1)V\varphi + MO_\infty(1)\mathcal{L}_\eta\varphi + M^2O_\infty(1)\mathcal{L}_\xi\varphi. \tag{4.8}$$

Proof. These follow by direct computation. □

Lemma 4.13. *Let b_ϕ, b_0 be conformally regular functions, let b_V be such that Rb_V is conformally regular, and let ϑ be a conformally regular spin-weighted scalar. If φ is a solution of*

$$\widehat{\boxdot}_s\varphi + b_VV\varphi + b_\phi\mathcal{L}_\eta\varphi + b_0\varphi = \vartheta, \tag{4.9}$$

and if the initial data for φ on Σ_{init} is smooth and compactly supported, then φ is conformally regular.

Proof. The essence of this proof is to apply standard local well-posedness results for linear wave equations in both the hyperboloidal coordinates (t,r,ω) and the compactified hyperboloidal coordinates $(x^a)=(t,R,\omega)$. Working in the compactified coordinate system, one finds

$$\begin{aligned}\widehat{\boxdot}_s(\varphi) ={}& (4C_{\text{hyp}}+MO_\infty(R))M^2\partial_t\partial_t\varphi + (-2+M^2O_\infty(R^2))\partial_t\partial_R\varphi\\ &+(4a+M^2O_\infty(R))\partial_t\partial_\phi\varphi + O_\infty(R^2)\partial_R\partial_R\varphi + MO_\infty(R^2)\partial_R\partial_\phi\varphi\\ &+(-2R+MO_\infty(R^2))\partial_R\varphi + (2aR+M^2O_\infty(R^2))\partial_\phi\varphi\\ &+(4C_{\text{hyp}}R+MO_\infty(R^2))M^2\partial_t\varphi + (2MR+M^2O_\infty(R^2))\varphi - S_s(\varphi),\end{aligned} \tag{4.10}$$

where S_s is given by (2.36c). The principal part of $\widehat{\boxdot}_s$ can be written

$$h^{ab}\partial_{x^a}\partial_{x^b} = h_0^{ab}\partial_{x^a}\partial_{x^b} + Rh_1^{ab}\partial_{x^a}\partial_{x^b} \tag{4.11}$$

where

$$h_0^{ab}\partial_{x^a}\partial_{x^b} = (4C_{\text{hyp}}M^2 - a^2\sin^2\theta)\partial_t\partial_t - 2\partial_t\partial_R + 4a\partial_t\partial_\phi - \partial_\theta\partial_\theta - \sin^{-2}\theta\partial_\phi\partial_\phi \tag{4.12}$$

and h_1^{ab} has conformally regular components. One finds that h_{ab} extends as a Lorentzian metric across $\mathscr{I}^+$ and that the level sets of t are spacelike with respect to h_{ab}.

The lower-order terms $b_\phi\mathcal{L}_\eta + b_0$ in (4.9) are conformally regular. Further, in view of (4.8), we have that b_VV is conformally regular. Thus, the operator on the left-hand side of (4.9) is conformally regular and has principal part with symbol given by the inverse conformal metric h^{ab}. Thus, equation (4.9) is a spin-weighted wave equation in the extended spacetime.

Since the initial data for φ is assumed to be compactly supported, there is some t and a smooth, spacelike surface Σ in the extended spacetime, which agrees with Σ_t for large r, such that φ is smooth and compactly supported on $\Sigma\cap\{R>0\}$, and such that the future domain of dependence of Σ includes $\mathscr{I}^+_{t,\infty} = \{R=0\}\cap(t,\infty)$. It follows that φ is smooth in the domain of dependence of Σ in $\mathbb{R}\times(-\epsilon, r_+^{-1})\times S^2$ with inverse metric h^{ab}, and in particular conformally regular. □

4.3. Norms.

Definition 4.14. Let φ be a spin-weighted scalar. Its norm is defined to be

$$|\varphi|^2 = \bar\varphi\varphi. \tag{4.13}$$

If φ is a spin-weighted scalar, then $|\varphi|^2 = \bar\varphi\varphi$ has GHP type $\{0,0\}$. It follows that $|\varphi|$ and expressions like $\nabla_a|\varphi|^2$ have an invariant sense, and we may use this fact to define Sobolev type norms on spaces of boost-weight zero scalars.

Definition 4.15. Let $n\geq 1$ be an integer, and let $\mathbb{X}=\{X_1,\dots,X_n\}$ be spin-weighted operators. Define a multi-index to be either the empty set or an ordered set $\mathbf{a}=(a_1,\dots,a_m)$ with $m\in\mathbb{Z}^+$ and $a_i\in\{1,\dots,n\}$ for $i\in\{1,\dots,m\}$. If $\mathbf{a}=\emptyset$, define $|\mathbf{a}|=0$ and define $\mathbb{X}^{\mathbf{a}}$ to be the identity operator. If $\mathbf{a}=(a_1,\dots,a_m)$, define $|\mathbf{a}|=m$ and define the operator

$$\mathbb{X}^{\mathbf{a}} = X_{a_1}X_{a_2}\dots X_{a_m}. \tag{4.14}$$

Definition 4.16. Let $\mathbb{X}$ be a set of spin-weighted first-order operators, and let φ be a spin-weighted scalar. For $k \in \mathbb{N}$, we define the order k pointwise norm

$$|\varphi|^2_{k,\mathbb{X}} = \sum_{|\mathbf{a}|\leq k} |\mathbb{X}^{\mathbf{a}}\varphi|^2. \tag{4.15}$$

Having defined norms in terms of sets of operators, we now introduce the following sets of operators. The operators in $\mathbb{B}$ have dimensions M^{-1} as is standard for derivative operators. The operators in the remaining sets have been scaled so that they are dimensionless.

Definition 4.17. Define

$$\mathbb{B} = \{Y, V, r^{-1}\mathring{\eth}, r^{-1}\mathring{\eth}'\}, \tag{4.16a}$$

$$\mathbb{D} = \{MY, rV, \mathring{\eth}, \mathring{\eth}'\}, \tag{4.16b}$$

$$\mathbb{S} = \{\mathring{\eth}, \mathring{\eth}'\}, \tag{4.16c}$$

$$\not{\mathbb{D}} = \{\mathring{\eth}, \mathring{\eth}', M\mathcal{L}_\xi\}. \tag{4.16d}$$

The following definition introduces weighted Sobolev spaces. Because the mass M provides a natural length scale, we are able to ensure that the integrands in the weighted Sobolev norms are dimensionless.

Definition 4.18. Let φ be a spin-weighted scalar. Let Ω denote a four-dimensional subset of the domain of outer communication, and let Σ denote a hypersurface in the domain of outer communication that can be parametrized by (r, ω). For an $k \in \mathbb{N}$ and $\gamma \in \mathbb{R}$, define

$$\|\varphi\|^2_{W^k_\gamma(\Omega)} = \int_\Omega M^{-\gamma-2} r^\gamma |\varphi|^2_{k,\mathbb{D}} \mathrm{d}^4\mu, \tag{4.17a}$$

$$\|\varphi\|^2_{W^k_\gamma(\Sigma)} = \int_\Sigma M^{-\gamma-1} r^\gamma |\varphi|^2_{k,\mathbb{D}} \mathrm{d}^3\mu, \tag{4.17b}$$

$$\|\varphi\|^2_{W^k(S^2)} = \int_{S^2} |\varphi|^2_{k,\mathbb{S}} \mathrm{d}^2\mu. \tag{4.17c}$$

We shall refer to norms $\|\varphi\|_{W^k_\gamma(\Omega_{t_1,t_2})}$ and $\|\varphi\|_{W^k_\gamma(\Sigma_t)}$ as weighted Morawetz and energy norms, respectively. We say that $\varphi \in W^k_\gamma(\Omega_{t_1,t_2})$ if $\|\varphi\|_{W^k_\gamma(\Omega_{t_1,t_2})} < \infty$ and similarly for $W^k_\gamma(\Sigma_t)$, $W^k_\gamma(\Xi_{t,\infty})$, $W^k(S^2)$, and so on.

Remark 4.19. In view of remark 2.1, the spaces $W^k_\gamma(\Omega), W^k_\gamma(\Sigma), W^k(S^2)$ etc., are Sobolev spaces of sections of Riemannian vector bundles, and by remark 2.18, when restricting to the sphere S^2, the operators $\mathring{\eth}, \mathring{\eth}'$ are elliptic operators of order one, acting on sections of these bundles. In the following we shall freely make use of these facts.

Definition 4.20. Let φ be a spin-weighted scalar, and let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$.

(1) Let Σ denote a hypersurface in the domain of outer communication that can be parametrized by (r, ω). Define

$$\|\varphi\|^2_{H^k_\alpha(\Sigma)} = \sum_{|\mathbf{a}|\leq k} \int_\Sigma M^{-\alpha} r^{\alpha+2|\mathbf{a}|-1} |\mathbb{B}^{\mathbf{a}}\varphi|^2 \mathrm{d}^3\mu \tag{4.18}$$

and introduce the quantity

$$\mathbb{I}^{k;\alpha}_{\text{init}}(\varphi) = \|\varphi\|^2_{H^k_\alpha(\Sigma_{\text{init}})}. \tag{4.19}$$

(2) Define the following norm on the surface Σ_{init}

$$\mathbb{P}^{k;\alpha}_{\text{init}}(\varphi) = \sup_{r\in[r_+,\infty)} \sum_{|\mathbf{a}|\leq k} M^{-\alpha} r^{\alpha+2|\mathbf{a}|} \int_{S^2} |\mathbb{B}^{\mathbf{a}}\varphi(t_0 - h(r)/2, r, \omega)|^2 \mathrm{d}^2\mu. \tag{4.20}$$

Remark 4.21. We have

$$\|\varphi\|_{W^k_\alpha(\Sigma_{\text{init}})} \lesssim \|\varphi\|_{H^k_{\alpha+1}(\Sigma_{\text{init}})}. \tag{4.21}$$

Definition 4.22. Let φ be a spin-weighted scalar, and let $k \in \mathbb{N}$. Define

$$\|\varphi\|^2_{F^k(\mathscr{I}^+_{-\infty,t})} = \int_{\mathscr{I}^+_{-\infty,t}} M|Y\varphi|^2_{k,\mathbb{D}} \mathrm{d}^3\mu_{\mathscr{I}}. \tag{4.22}$$

4.4. **Basic estimates.** The operators $\mathring{\eth}, \mathring{\eth}'$ are the spherical edth operators, cf. [19] for background. In particular, they are elliptic first order operators acting on properly weighted functions on the sphere. For completeness, we recall some useful facts about $\mathring{\eth}, \mathring{\eth}'$.

Lemma 4.23. *Let φ, ψ be scalars with spin-weight s and $s-1$ respectively. Then,*

(1)

$$\int \psi \mathring{\eth} \varphi \mathrm{d}^2\mu = \int (\mathring{\eth}' \psi)\varphi \mathrm{d}^2\mu. \tag{4.23}$$

(2) if $s=-1$ it holds that

$$\int_{S^2} \mathring{\eth} \varphi \mathrm{d}^2\mu = 0; \tag{4.24}$$

(3) if $s=1$, it holds that

$$\int_{S^2} \mathring{\eth}' \varphi \mathrm{d}^2\mu = 0; \tag{4.25}$$

(4) we have the following relation between $\|\mathring{\eth}\varphi\|_{L^2(S^2)}$ and $\|\mathring{\eth}'\varphi\|_{L^2(S^2)}$:

$$\int_{S^2} |\mathring{\eth}\varphi|^2 \mathrm{d}^2\mu = \int_{S^2} |\mathring{\eth}'\varphi|^2 \mathrm{d}^2\mu - s\int_{S^2} |\varphi|^2 \mathrm{d}^2\mu. \tag{4.26}$$

Proof. The first claim appears in [31]. The second follows from taking $\psi = 1$, and the third follows from complex conjugation. For the fourth claim, we multiply both sides of the commutator relation (2.42d) by $\bar{\varphi}$ and use the Leibniz rule to obtain

$$\mathring{\eth}(\mathring{\eth}'\varphi\bar{\varphi}) - \mathring{\eth}\bar{\varphi}\mathring{\eth}'\varphi = \mathring{\eth}'(\mathring{\eth}\varphi\bar{\varphi}) - \mathring{\eth}'\bar{\varphi}\mathring{\eth}\varphi - s|\varphi|^2. \tag{4.27}$$

By integrating over S^2 and noting the facts that $\mathring{\eth}'\varphi\bar{\varphi}$ has boost- and spin-weight $0,-1$ and $\mathring{\eth}\varphi\bar{\varphi}$ has boost- and spin-weight $0,1$, the integrals over S^2 of the first term on the left and the first term on the right are both vanishing, hence the relation (4.26) follows. □

Lemma 4.24. *Let φ be a scalar of spin-weight s. For any $k \geq 0$, it holds*

$$\int_{S^2} |\varphi|^2_{2k,\mathbb{S}} \mathrm{d}^2\mu \sim_s \sum_{i=0}^{k} \int_{S^2} |\mathring{S}^i_s\varphi|^2 \mathrm{d}^2\mu. \tag{4.28}$$

Proof. Since $\mathring{\eth}$ and $\mathring{\eth}'$ are both in $\mathbb{S}$, this follows from the relation (2.36d) and the fact that $\mathring{\eth}, \mathring{\eth}'$ are elliptic operators of order one [30, Theorem III.5.2]. □

Lemma 4.25 (Eigenvalue estimates for $\mathring{\eth}, \mathring{\eth}'$)**.** *If φ is a scalar of spin-weight s, then*

$$\frac{|s|-s}{2}\int_{S^2} |\varphi|^2 \mathrm{d}^2\mu \leq \int_{S^2} |\mathring{\eth}\varphi|^2 \mathrm{d}^2\mu, \tag{4.29a}$$

$$\frac{|s|+s}{2}\int_{S^2} |\varphi|^2 \mathrm{d}^2\mu \leq \int_{S^2} |\mathring{\eth}'\varphi|^2 \mathrm{d}^2\mu, \tag{4.29b}$$

and for a four dimensional spacetime region Ω,

$$\frac{|s|-s}{2}\|\varphi\|^2_{W^k_\gamma(\Omega)} \leq \|\mathring{\eth}\varphi\|^2_{W^k_\gamma(\Omega)}, \tag{4.30a}$$

$$\frac{|s|+s}{2}\|\varphi\|^2_{W^k_\gamma(\Omega)} \leq \|\mathring{\eth}'\varphi\|^2_{W^k_\gamma(\Omega)}. \tag{4.30b}$$

The first (second) case gives an estimate if φ has negative (positive) spin-weight.

Proof. We will prove the statement for $\mathring{\eth}$. The statement for $\mathring{\eth}'$ follows by complex conjugation. Expand φ in terms of spin-weighted spherical harmonics (see [38, Section 4.15])

$$\varphi(\theta,\phi) = \sum_{l=|s|}^{\infty}\sum_{m=-l}^{l} a_{l,m}\, {}_sY_{lm}(\theta,\phi). \tag{4.31}$$

From [38, Eq. (4.15.106)] we have

$$\mathring{\eth}\,\varphi(\theta,\phi) = -\sum_{l=|s|}^{\infty}\sum_{m=-l}^{l} a_{l,m}\frac{\sqrt{(l+s+1)(l-s)}}{\sqrt{2}}\,{}_{s+1}Y_{lm}(\theta,\phi). \tag{4.32}$$

Through the orthogonality conditions [38, Eq. (4.15.99)] we get

$$\int_{S^2}|\varphi|^2\mathrm{d}^2\mu = 4\pi\sum_{l=|s|}^{\infty}\sum_{m=-l}^{l}|a_{l,m}|^2, \tag{4.33a}$$

$$\int_{S^2}|\mathring{\eth}\,\varphi|^2\mathrm{d}^2\mu = 4\pi\sum_{l=|s|}^{\infty}\sum_{m=-l}^{l}|a_{l,m}|^2\frac{(l+s+1)(l-s)}{2}. \tag{4.33b}$$

As $(l+s+1)(l-s)\geq |s|-s$, this proves (4.29a). Integrating in t, r gives the remaining results. $\square$

Lemma 4.26 (Control of $\mathcal{L}_\eta$ in $L^2(S^2)$)**.** *If φ is a scalar of spin weight s, then*

$$\frac{1}{2}\int_{S^2}|\mathcal{L}_\eta\varphi|^2\mathrm{d}^2\mu \leq \int_{S^2}\left(|\mathring{\eth}\,\varphi|^2+\frac{s^2}{2}|\varphi|^2\right)\mathrm{d}^2\mu.$$

Proof. This follows from decomposing into spin-weighted spherical harmonics ${}_sY_{l,m}$, the relations $|m|\leq l$ and $|s|\leq l$, from equations (4.33a)-(4.33b), and the fact that $(l+s+1)(l-s)+s^2 = l^2+l-s\geq l^2\geq m^2$. $\square$

Lemma 4.27 (Spherical Sobolev estimate)**.** *If φ is a scalar of spin-weight s, then*

$$|\varphi|^2 \lesssim_s \int_{S^2}|\varphi|^2_{2,\mathbb{S}}\mathrm{d}^2\mu. \tag{4.34}$$

Proof. The right-hand side of (4.34) is the norm on the space $W^2(S^2)$. The standard Sobolev estimate for sections of vector bundles applies. See [30, Theorems III.2.15 and II.5.2]. $\square$

Lemma 4.28 (Integration by parts)**.** *If f is a smooth scalar with spin- and boost-weight zero and if f vanishes at R_0, then*

$$\int_{\Omega^{R_0}_{t_1,t_2}} Yf\mathrm{d}^4\mu = \left[\int_{\Sigma^{R_0}_t}(\mathrm{d}t_aY^a)f\mathrm{d}^3\mu\right]_{t=t_1}^{t_2}, \tag{4.35a}$$

$$\int_{\Omega^{R_0}_{t_1,t_2}} Vf\mathrm{d}^4\mu = \left[\int_{\Sigma^{R_0}_t}f(\mathrm{d}t_aV^a)\mathrm{d}^3\mu\right]_{t=t_1}^{t_2} - \int_{\Omega^{R_0}_{t_1,t_2}}M\frac{r^2-a^2}{(r^2+a^2)^2}f\mathrm{d}^4\mu+\frac{1}{2}\int_{\mathscr{I}^+_{t_1,t_2}}f\mathrm{d}^3\mu_{\mathscr{I}}. \tag{4.35b}$$

Proof. In ingoing Eddington-Finkelstein coordinates, $\mathrm{d}^4\mu = \sin\theta\mathrm{d}\phi\mathrm{d}\theta\mathrm{d}r\mathrm{d}v$. The first claim follows from the fact that Y is $-\partial_r$ in ingoing Eddington-Finkelstein coordinates. The second claim follows from equation (2.40a) and that

$$\partial_r\left(\frac{\Delta}{2(r^2+a^2)}\right) = M\frac{r^2-a^2}{(r^2+a^2)^2}. \tag{4.36}$$

$\square$

Lemma 4.29 (Weighted integration by parts)**.** *Let f be a smooth, real-valued function of r and θ that vanishes at R_0 and φ be a spin-weighted scalar.*

$$\int_{\Omega^{R_0}_{t_1,t_2}} \Re\left(f\bar{\varphi}Y\varphi\right)\mathrm{d}^4\mu = \left[\int_{\Sigma^{R_0}_t} (\mathrm{d}t_a Y^a)\frac{1}{2}f|\varphi|^2\mathrm{d}^3\mu\right]_{t=t_1}^{t_2} + \int_{\Omega^{R_0}_{t_1,t_2}} \frac{1}{2}(\partial_r f)|\varphi|^2\mathrm{d}^4\mu, \tag{4.37a}$$

$$\int_{\Omega^{R_0}_{t_1,t_2}} \Re\left(f\bar{\varphi}V\varphi\right)\mathrm{d}^4\mu = \left[\int_{\Sigma^{R_0}_t} (\mathrm{d}t_a V^a)\frac{1}{2}f|\varphi|^2\mathrm{d}^3\mu\right]_{t=t_1}^{t_2} - \int_{\Omega^{R_0}_{t_1,t_2}} \partial_r\left(f\frac{\Delta}{4(r^2+a^2)}\right)|\varphi|^2\mathrm{d}^4\mu$$
$$+\frac{1}{4}\int_{\mathscr{I}^+_{t_1,t_2}} f|\varphi|^2\mathrm{d}^3\mu_{\mathscr{I}}. \tag{4.37b}$$

Proof. This follows from the previous lemma and the fact that $\Re(f\bar{\varphi}V\varphi) = V\left(\frac{1}{2}f|\varphi|^2\right) - (Vf)|\varphi|^2/2$, and similarly for Y. ☐

The following lemma gives a standard one-dimensional Hardy inequality. The subsequent lemma applies this to obtain a similar estimate on each $\Sigma_t^{R_0-M}$ with an estimate in terms of the operators V and Y.

Lemma 4.30 (One-dimensional Hardy estimates)**.** *Let $\gamma \in \mathbb{R}\setminus\{0\}$ and $h : [r_0, r_1] \to \mathbb{R}$ be a C^1 function.*

(1) If $r_0^\gamma|h(r_0)|^2 \le D_0$ and $\gamma < 0$, then

$$-2\gamma^{-1}r_1^\gamma|h(r_1)|^2 + \int_{r_0}^{r_1} r^{\gamma-1}|h(r)|^2\mathrm{d}r \le \frac{4}{\gamma^2}\int_{r_0}^{r_1} r^{\gamma+1}|\partial_r h(r)|^2\mathrm{d}r - 2\gamma^{-1}D_0. \tag{4.38a}$$

(2) If $r_1^\gamma|h(r_1)|^2 \le D_0$ and $\gamma > 0$, then

$$2\gamma^{-1}r_0^\gamma|h(r_0)|^2 + \int_{r_0}^{r_1} r^{\gamma-1}|h(r)|^2\mathrm{d}r \le \frac{4}{\gamma^2}\int_{r_0}^{r_1} r^{\gamma+1}|\partial_r h(r)|^2\mathrm{d}r + 2\gamma^{-1}D_0. \tag{4.38b}$$

Proof. We integrate $\partial_r(r^\gamma|h|^2)$ over $[r_0, r_1]$ to obtain:

$$r_1^\gamma|h(r_1)|^2 - r_0^\gamma|h(r_0)|^2 = \gamma\int_{r_0}^{r_1} r^{\gamma-1}|h(r)|^2\mathrm{d}r + 2\int_{r_0}^{r_1} r^\gamma\Re\{\bar{h}\partial_r h\}\mathrm{d}r. \tag{4.39}$$

In the first case where $\gamma < 0$, we apply a Cauchy-Schwarz inequality to estimate the last integral term

$$\left|2\int_{r_0}^{r_1} r^\gamma\Re\{\bar{h}\partial_r h\}\mathrm{d}r\right| \le \frac{-\gamma}{2}\int_{r_0}^{r_1} r^{\gamma-1}|h(r)|^2\mathrm{d}r + \frac{2}{-\gamma}\int_{r_0}^{r_1} r^{\gamma+1}|\partial_r h(r)|^2\mathrm{d}r \tag{4.40}$$

Collecting the above two estimates implies (4.38a). The estimate (4.38b) follows in the same way. ☐

Lemma 4.31 (Hardy estimate on hypersurfaces)**.** *Let $\varepsilon > 0$. There is an $\bar{R}_0 \ge 10M$ such that for $R_0 \ge \bar{R}_0$ and all spin-weighted scalars φ,*

$$\|\varphi\|^2_{W^0_{-2}(\Sigma_t^{R_0-M})} \le (16+\varepsilon)\|rV\varphi\|^2_{W^0_{-2}(\Sigma_t^{R_0-M})} + \varepsilon\|MY\varphi\|^2_{W^0_{-2}(\Sigma_t^{R_0-M})} + \|\varphi\|^2_{W^0_0(\Sigma_t^{R_0-M,R_0})}. \tag{4.41}$$

Similarly for $\delta > 0$ and $\alpha \in [\delta, 2-\delta]$, there is a constant $\bar{R}_0 = \bar{R}_0(\delta) \ge 10M$ such that for $R_0 \ge \bar{R}_0$ and all spin-weighted scalars φ,

$$\|\varphi\|^2_{W^0_{\alpha-3}(\Sigma_t^{R_0-M})} \lesssim \|rV\varphi\|^2_{W^0_{\alpha-3}(\Sigma_t^{R_0-M})} + \|MY\varphi\|^2_{W^0_{-\delta-1}(\Sigma_t^{R_0-M})} + \|\varphi\|^2_{W^0_0(\Sigma_t^{R_0-M,R_0})}. \tag{4.42}$$

Proof. Let

$$\mathcal{X} = h'V + (1 - \Delta h'/(2(r^2+a^2)))Y. \tag{4.43}$$

Then the vector field $\mathcal{X}^a$ corresponding to $\mathcal{X}$ is tangent to Σ_t. We may introduce new coordinates $(\tilde{r}, \tilde{\theta}, \tilde{\phi})$ on Σ_t by taking $\tilde{r} = r$, $\tilde{\theta} = \theta$, and $\tilde{\phi}$ is constant along the flow lines of $\mathcal{X}^a$ such $\tilde{\phi}$ agrees with ϕ on $r = R_0$. In such coordinates and the Znajek tetrad, one finds that $\mathcal{X}$ is $\partial_{\tilde{r}}$.

From the one-dimensional Hardy estimate (4.38a) with $\gamma = -1$, one finds for sufficiently large r

$$\int_r^\infty (r')^{-2}|\varphi(r',\omega)|^2 \mathrm{d}r' \leq 4\int_r^\infty |\mathfrak{X}\varphi(r',\omega)|^2 \mathrm{d}r' + 2r^{-1}|\varphi(r,\omega)|^2. \tag{4.44}$$

Integrating this over $r \in (R_0 - M, R_0)$, and since $R_0 \geq 10M$, one finds

$$M\int_{R_0}^\infty r^{-2}|\varphi|^2 \mathrm{d}^3\mu \leq 4M\int_{R_0-M}^\infty |\mathfrak{X}\varphi|^2 \mathrm{d}^3\mu + 4MR_0^{-1}\int_{\Sigma_t^{R_0-M,R_0}} |\varphi|^2 \mathrm{d}^3\mu. \tag{4.45}$$

From the definition of $\mathfrak{X}$ in equation (4.43), the expansion for h' in equation (2.48), and the observation that the Y coefficient in $\mathfrak{X}$ satisfies

$$1 - \frac{\Delta h'}{2(r^2+a^2)} = M^2 O_\infty(r^{-2}), \tag{4.46}$$

it follows that for sufficiently large r, there is the bound $4|\mathfrak{X}\varphi|^2 \leq (16+\varepsilon)|V\varphi|^2 + \varepsilon M^2 r^{-2}|Y\varphi|^2$, which completes the proof.

For $\alpha \in [\delta, 2-\delta]$, a similar argument applies, except the bound $\alpha - 3 \leq -\delta - 1$ is used. The constant in the one-dimensional Hardy estimate (4.38a) diverges as $\gamma = \alpha - 2$ goes to zero, but, if α is restricted to an interval $[\delta, 2-\delta]$ the constant is uniform in α, but depends upon δ. □

Lemma 4.32 (Sobolev estimate on hypersurfaces)**.** *Assume φ is a scalar of spin-weight s, and let $\mathfrak{X}$ be the operator from the proof of the Hardy lemma 4.31. For $\gamma \in \mathbb{R}$, we have, for $t \geq t_0$,*

$$\begin{aligned}\sup_{\Sigma_t}|\varphi|^2 &\lesssim_s \left(\int_{\Sigma_t} r^{-1-\gamma}|\varphi|^2_{2,\mathbb{S}}\mathrm{d}^3\mu \int_{\Sigma_t} r^{-1+\gamma}|r\mathfrak{X}\varphi|^2_{2,\mathbb{S}}\mathrm{d}^3\mu\right)^{1/2} + \int_{\Sigma_t^{r_+,10M}} M^{-1}|\varphi|^2_{2,\mathbb{S}}\mathrm{d}^3\mu \\ &\lesssim_s \|\varphi\|_{W^2_{-1-\gamma}(\Sigma_t)}\|r\mathfrak{X}\varphi\|_{W^2_{-1+\gamma}(\Sigma_t)} + \|\varphi\|_{W^2_0(\Sigma_t^{r_+,10M})}.\end{aligned} \tag{4.47}$$

In the case that $\gamma = 0$, we have

$$\sup_{\Sigma_t}|\varphi|^2 \lesssim_s \|\varphi\|^2_{W^3_{-1}(\Sigma_t)}. \tag{4.48}$$

If $0 < \gamma \leq 1$, we also have

$$\sup_{\Sigma_t}|\varphi|^2 \lesssim_{\gamma,s} (\|\varphi\|^2_{W^3_{-2}(\Sigma_t)} + \|rV\varphi\|^2_{W^2_{-1+\gamma}(\Sigma_t)})^{1/2}(\|\varphi\|^2_{W^3_{-2}(\Sigma_t)} + \|rV\varphi\|^2_{W^2_{-1-\gamma}(\Sigma_t)})^{1/2}. \tag{4.49}$$

Proof. Let $\mathfrak{X}$ be as in the proof of lemma 4.31. For $r_1, r_2 \in [r_+, \infty)$, one has

$$\begin{aligned}\int_{S^2}|\varphi(r_2)|^2\mathrm{d}^2\mu &= \left|\int_{r_1}^{r_2}\int_{S^2}\partial_r|\varphi(r)|^2\mathrm{d}^3\mu\right| + \int_{S^2}|\varphi(r_1)|^2\mathrm{d}^2\mu, \\ &= \left|\int_{r_1}^{r_2}\int_{S^2}\mathfrak{X}|\varphi(r)|^2\mathrm{d}^3\mu\right| + \int_{S^2}|\varphi(r_1)|^2\mathrm{d}^2\mu, \\ &\leq \left(\int_{r_1}^\infty\int_{S^2} r^{-1-\gamma}|\varphi(r)|^2\mathrm{d}^3\mu\int_{r_1}^\infty\int_{S^2} r^{-1+\gamma}|r\mathfrak{X}\varphi(r)|^2\mathrm{d}^3\mu\right)^{1/2} + \int_{S^2}|\varphi(r_1)|^2\mathrm{d}^2\mu,\end{aligned} \tag{4.50}$$

where in the last step we have used Hölder inequality. We integrate over r_1 from r_+ to $10M$ and the first line of (4.47) holds from the spherical Sobolev lemma 4.27 where the integral is taken to be over the sphere with given t and r. The second line of (4.47) holds since $\mathbb{S} \subset \mathbb{D}$.

The estimate (4.48) when $\gamma = 0$ follows from applying Cauchy-Schwarz inequality to the right of (4.47) and the fact that $r\mathfrak{X}$ is in the span of rV and $M^2r^{-1}Y$ with $O_\infty(1)$ coefficients.

We now prove the estimate (4.49). Since $\gamma > 0$, one can use the Hardy inequality (4.38a) to arrive at

$$\int_{r_+}^\infty\int_{S^2} r^{-1-\gamma}|\varphi(r)|^2\mathrm{d}^3\mu \lesssim_\gamma \int_{r_+}^\infty\int_{S^2} r^{-1-\gamma}|r\mathfrak{X}\varphi(r)|^2\mathrm{d}^3\mu + \int_{r_+}^{10M}\int_{S^2} r^{-1-\gamma}|\varphi(r)|^2\mathrm{d}^3\mu. \tag{4.51}$$

Hence, by integrating (4.50) over r_1 from r_+ to $10M$, one finds for any $r \in [r_+, \infty)$

$$\int_{S^2} |\varphi(r)|^2 \mathrm{d}^2\mu \lesssim_\gamma \left(\int_{r_+}^{\infty} \int_{S^2} r^{-1-\gamma} |r\mathcal{X}\varphi(r)|^2 \mathrm{d}^3\mu \int_{r_+}^{\infty} \int_{S^2} r^{-1+\gamma} |r\mathcal{X}\varphi(r)|^2 \mathrm{d}^3\mu \right)^{1/2} + \int_{r_+}^{10M} \int_{S^2} M^{-1} |\varphi(r)|^2 \mathrm{d}^3\mu. \tag{4.52}$$

Since $r\mathcal{X}$ is in the span of rV and $M^2 r^{-1} Y$ with $O_\infty(1)$ coefficients, and since $\mathbb{S} \subset \mathbb{D}$ and the assumption $\gamma \leq 1$, the estimate (4.49) then follows. □

Lemma 4.33 (Anisotropic, spacetime Sobolev inequality). *Let φ be a scalar of spin-weight s. If $\lim_{t\to\infty} |r^{-1}\varphi| = 0$ pointwise in (r, ω), then*

$$|r^{-1}\varphi|^2 \lesssim_s \|\varphi\|_{W^3_{-3}(\Omega_{t,\infty})} \|\mathcal{L}_\xi \varphi\|_{W^3_{-3}(\Omega_{t,\infty})}. \tag{4.53}$$

Proof. Using the fundamental theorem of calculus and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |r^{-1}\varphi|^2 &= -\int_t^\infty \mathcal{L}_\xi |r^{-1}\varphi|^2 \mathrm{d}t' \\ &\leq 2 \int_t^\infty |\mathcal{L}_\xi r^{-1}\varphi| |r^{-1}\varphi| \mathrm{d}t' \\ &\leq 2 \left(\int_t^\infty |r^{-1}\mathcal{L}_\xi \varphi|^2 \mathrm{d}t' \right)^{1/2} \left(\int_t^\infty |r^{-1}\varphi|^2 \mathrm{d}t' \right)^{1/2}. \end{aligned} \tag{4.54}$$

Now, from applying the Sobolev inequality (4.48) on each Σ_t, the result holds. □

Lemma 4.34 (Transition flux is controlled by bulk). *Let $f(t, r)$ be a spin-weighted scalar. For any real value γ and $t \geq t_0 \geq 1$, it holds true that*

$$\int_t^\infty (t')^\gamma |f(t', t')|^2 \mathrm{d}t' \lesssim_\gamma \int_t^\infty \int_{t'}^\infty r^{\gamma-1} (|f(t', r)|^2 + |r\mathcal{X} f(t', r)|^2) \mathrm{d}r \mathrm{d}t'. \tag{4.55}$$

Proof. We make a change of coordinate $r = t' + \zeta$. Fix any $t'' \geq t$. From the mean-value principle, we can find a $\zeta' \in [t'', 2t'']$ such that

$$\begin{aligned} \int_t^{t+t''} (t' + \zeta')^{\gamma-1} |f(t', t' + \zeta')|^2 \mathrm{d}t' &\leq (t'')^{-1} \int_{t''}^{2t''} \int_t^{t+t''} (t' + \zeta)^{\gamma-1} |f(t', t' + \zeta)|^2 \mathrm{d}t' \mathrm{d}\zeta \\ &\leq (t'')^{-1} \int_t^\infty \int_0^\infty (t' + \zeta)^{\gamma-1} |f(t', t' + \zeta)|^2 \mathrm{d}\zeta \mathrm{d}t'. \end{aligned} \tag{4.56}$$

Therefore, for the ζ' chosen above,

$$\begin{aligned} \int_t^{t+t''} (t' + \zeta')^\gamma |f(t', t' + \zeta')|^2 \mathrm{d}t' &\leq 4t'' \int_t^{t+t''} (t' + \zeta')^{\gamma-1} |f(t', t' + \zeta')|^2 \mathrm{d}t' \\ &\leq 4 \int_t^\infty \int_0^\infty (t' + \zeta)^{\gamma-1} |f(t', t' + \zeta)|^2 \mathrm{d}\zeta \mathrm{d}t' \\ &\leq 4 \int_t^\infty \int_{t'}^\infty r^{\gamma-1} |f(t', r)|^2 \mathrm{d}r \mathrm{d}t'. \end{aligned} \tag{4.57}$$

Since $t'' \geq t$, $t' \in [t, t + t'']$ and $\zeta' \in [t'', 2t'']$, we have $t' + \zeta' \in [t', 4t'']$. It then follows from the fundamental theorem of calculus that

$$\begin{aligned} &\int_t^{t+t''} (t')^\gamma |f(t', t')|^2 \mathrm{d}t' \\ &\leq \int_t^{t+t''} (t' + \zeta')^\gamma |f(t', t' + \zeta')|^2 \mathrm{d}t' + \int_t^{t+t''} \int_{t'}^{4t''} |\mathcal{X}(r^\gamma |f(t', r)|^2)| \mathrm{d}r \mathrm{d}t' \\ &\leq C(\gamma) \int_t^\infty \int_{t'}^\infty r^{\gamma-1} (|f(t', r)|^2 + |r\mathcal{X}(f(t', r))|^2) \mathrm{d}r \mathrm{d}t'. \end{aligned} \tag{4.58}$$

Letting t'' go to infinity proves the estimate (4.55). □

Lemma 4.35 (Taylor expansion in L^2)**.** *Let $A > 0$, $n \in \mathbb{N}$, $f \in C^{n+1}([0, A])$, and*

$$P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} f^{(k)}(0). \tag{4.59}$$

Then for any $-1 < \alpha < 1$, there exists a constant $C = C(n, \alpha)$ such that

$$\Big\| \frac{f(x) - P_n(x)}{x^{n+1+\alpha/2}} \Big\|_{L^2((0,A))} \leq C \| x^{-\alpha/2} f^{(n+1)} \|_{L^2((0,A))}. \tag{4.60}$$

Proof. From the assumptions on the function $f(x)$, we have for any integer $0 \leq i \leq n$, there exist constants $C(n, i)$ such that

$$\lim_{x\to 0^+} \frac{\partial_x^i (f(x) - P_n(x))}{x^{n+1-i}} = C(n, i) f^{(n+1)}(0). \tag{4.61}$$

Given any integer $0 \leq i \leq n$, we do the replacements

$$(r, r_0, r_1, h(r), \gamma) \mapsto (x, 0, A, \partial_x^i (f(x) - P_n(x)), -2n + 2i - 1 - \alpha) \tag{4.62}$$

in point (1) of lemma 4.30, and note from the assumption $\alpha \in (-1, 1)$ and the fact (4.61) that

$$\gamma = -2n + 2i - 1 - \alpha < 0, \tag{4.63a}$$

$$\lim_{x\to 0^+} x^{-2n+2i-1-\alpha} (\partial_x^i (f(x) - P_n(x)))^2 = 0. \tag{4.63b}$$

Therefore, it follows from point (1) of lemma 4.30 that for any integer $0 \leq i \leq n$ and any $\alpha \in (-1, 1)$,

$$\int_0^A \frac{(\partial_x^i (f(x) - P_n(x)))^2}{x^{2n-2i+2+\alpha}} \mathrm{d}x \leq C(n, i, \alpha) \int_0^A \frac{(\partial_x^{i+1} (f(x) - P_n(x)))^2}{x^{2n-2i+\alpha}} \mathrm{d}x. \tag{4.64}$$

Thus, by induction, one finds, for $i \in \{0, \dots, n\}$, that

$$\| x^{-n-1-\alpha/2} (f - P_n) \|_{L^2((0,A))} \lesssim \| x^{-n+i-\alpha/2} \partial_x^{i+1} (f - P_n) \|_{L^2((0,A))}. \tag{4.65}$$

The case $i = n$ gives the desired result. □

Lemma 4.36. *For a spin-weighted scalar φ and for any $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, there is the bound*

$$\mathbb{P}^{k;\alpha}_{\text{init}}(\varphi) \lesssim_\alpha \mathbb{I}^{k+1;\alpha}_{\text{init}}(\varphi). \tag{4.66}$$

Proof. From the definition of $\mathbb{I}^{k+1;\alpha}_{\text{init}}$ and commuting r through the $\mathbb{B}$ derivatives, it follows that

$$\begin{aligned} \mathbb{I}^{k+1;\alpha}_{\text{init}}(\varphi) &= \sum_{|\mathbf{a}|\leq k+1} \int_{\Sigma_{\text{init}}} M^{-\alpha} r^{\alpha+2|\mathbf{a}|-1} |\mathbb{B}^{\mathbf{a}} \varphi|^2 \mathrm{d}^3\mu \\ &\gtrsim_\alpha \sum_{|\mathbf{a}|\leq k+1} \int_{\Sigma_{\text{init}}} M^{-\alpha} r^2 |\mathbb{B}^{\mathbf{a}} (r^{\alpha/2+|\mathbf{a}|-3/2} \varphi)|^2 \mathrm{d}^3\mu. \end{aligned} \tag{4.67}$$

There are two important consequences of this. First, one finds, from ignoring the case $|\mathbf{a}| = 0$ and the divergence of $\int r^{-1} \mathrm{d}r$, that

$$r \sum_{|\mathbf{a}|\leq k} \int_{S^2} M^{-\alpha} r^2 |\mathbb{B}^{\mathbf{a}} (r^{\alpha/2+|\mathbf{a}|-3/2} \varphi)|^2 \mathrm{d}^3\mu \to 0 \tag{4.68}$$

as $r \to \infty$, at least along some sequence. Before considering the second, observe that there is a vector field $\mathcal{X}^a$ that is parallel to Σ_{init} and the corresponding operator $\mathcal{X}$ has an expansion solely in terms of V and Y with $O_\infty(1)$ coefficients. As in lemma 4.31, this can be used to define a radial coordinate $\tilde{r}$ such that, in the Znajek tetrad, $\mathcal{X} = \partial_{\tilde{r}}$ on Σ_{init}. Thus, there is the second observation that

$$\mathbb{I}^{k+1;\alpha}_{\text{init}}(\varphi) \gtrsim_\alpha \sum_{|\mathbf{a}|\leq k} \int_{\Sigma_{\text{init}}} M^{-\alpha} r^2 |\mathcal{X} \mathbb{B}^{\mathbf{a}} (r^{\alpha/2+|\mathbf{a}|-1/2} \varphi)|^2 \mathrm{d}^3\mu. \tag{4.69}$$

where we have taken into account the shift in i. Now applying the pointwise control in point 2 of Lemma 4.30 with $\gamma = 1$, and using the limit (4.68) to drop the right endpoint, one concludes for any $(t, r, \omega) \in \Sigma_{\text{init}}$,

$$\begin{aligned}\mathbb{I}^{k+1;\alpha}_{\text{init}}(\varphi) &\gtrsim_\alpha r \sum_{|\mathbf{a}|\leq k} M^{-\alpha} \int_{S^2} |\mathbb{B}^{\mathbf{a}} r^{\alpha/2+|\mathbf{a}|-1/2} \varphi(t, r, \omega)|^2 \mathrm{d}^3\mu \\ &\gtrsim_\alpha \sum_{|\mathbf{a}|\leq k} M^{-\alpha} r^{\alpha+2|\mathbf{a}|} \int_{S^2} |\mathbb{B}^{\mathbf{a}} \varphi(t, r, \omega)|^2 \mathrm{d}^3\mu. \end{aligned} \tag{4.70}$$

By taking the supremum in $r \in [r_+, \infty)$ and $t = t_0 - h(r)/2$, this completes the proof. $\square$

5. Weighted energy estimates

5.1. **A hierarchy of pointwise and integral estimates implies decay.** This subsection provides some simple lemmas for treating hierarchies of decay estimates. Such hierarchies arise both in the analysis of the Teukolsky equation and in the analysis of transport equations. The The proof of these results relies on the (continuous) pigeonhole principle.

For transport equations, the hierarchy of estimates is generally fairly straightforward, with a weighted integral of a solution being controlled by a weighted integral of a source. However, for wave-like equations, such as the Teukolsky equation, one finds that the weighted integral of a function at one level of regularity is estimated in terms of another weighted integral at a different level of regularity. For this reason, lemma 5.2 involves a function $f(i', \alpha, t)$, which should be thought of as being an integral involving a regularity i', a weight α, and a time t.

The following lemma uses a single application of the pigeonhole principle and is used in the proof of lemma 5.2.

Lemma 5.1 (Single step)**.** *Let $f : \{-1, 0, 1\} \times [t_0, \infty) \to [0, \infty)$ be such that $f(i', t)$ is Lebesgue mesurable in t for each i'. If there is a $D \geq 0$ and $\alpha \in \mathbb{R}$ such that, for all $i' \in \{0, 1\}$ and $t_2 \geq t_1 \geq t_0$,*

$$f(i', t_2) + \int_{t_1}^{t_2} f(i' - 1, t)\mathrm{d}t \lesssim f(i', t_1) + t_1^{\alpha+i'} D, \tag{5.1}$$

then, for all $t \geq 2t_0$,

$$f(0, t) \lesssim_\alpha t^{-1} f(1, t/2) + t^\alpha D. \tag{5.2}$$

Proof. From the mean-value principle, for any $t \geq 2t_0$, there is a $\tilde{t} \in [t/2, t]$ such that

$$f(0, \tilde{t}) \leq \frac{2}{t} \int_{t/2}^{t} f(0, t')\mathrm{d}t'. \tag{5.3}$$

Combining this with the integral estimate for $i' = 1$ in hypothesis (5.1), one can control f at $\tilde{t}$ by

$$f(0, \tilde{t}) \lesssim 2t^{-1} \left(f(1, t/2) + (t/2)^{\alpha+1} D\right) \lesssim_\alpha t^{-1} f(1, t/2) + t^\alpha D. \tag{5.4}$$

From the pointwise estimate for $i' = 0$ in hypothesis (5.1), one can control f at t by

$$f(0, t) \lesssim f(0, \tilde{t}) + \tilde{t}^\alpha D \lesssim_\alpha f(0, \tilde{t}) + t^\alpha D. \tag{5.5}$$

The lemma follows from combining estimates (5.4) and (5.5). $\square$

The following lemma proves that a hierarchy of decay estimates implies a decay rate for the terms in the hierarchy. In applications, i' represents a level of regularity, α represents a weight, and t represents a time coordinate. The weights take values in an interval, whereas the levels of regularity are discrete.

Lemma 5.2 (A hierarchy of estimates implies decay rates)**.** *Let $D \geq 0$. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $i \in \mathbb{Z}^+$ be such that $\alpha_1 \leq \alpha_2 - 1$, and $\alpha_2 - \alpha_1 \leq i$. Let $F : \{-1, \ldots, i\} \times [\alpha_1 - 1, \alpha_2] \times [t_0, \infty) \to [0, \infty)$ be such that $F(i', \alpha, t)$ is Lebesgue measurable in t for each α and i'. Let $\gamma \geq 0$.*

If

(1) *[monotonicity] for all $i', i'_1, i'_2 \in \{-1, \dots, i\}$ with $i'_1 \le i'_2$, all $\beta, \beta_1, \beta_2 \in [\alpha_1, \alpha_2]$ with $\beta_1 \le \beta_2$, and all $t \ge t_0$,*

$$F(i'_1, \beta, t) \lesssim F(i'_2, \beta, t), \tag{5.6a}$$

$$F(i', \beta_1, t) \lesssim F(i', \beta_2, t), \tag{5.6b}$$

(2) *[interpolation] for all $i' \in \{-1, \dots, i\}$, all $\alpha, \beta_1, \beta_2 \in [\alpha_1, \alpha_2]$ such that $\beta_1 \le \alpha \le \beta_2$, and all $t \ge t_0$,*

$$F(i', \alpha, t) \lesssim F(i', \beta_1, t)^{\frac{\beta_2 - \alpha}{\beta_2 - \beta_1}} F(i', \beta_2, t)^{\frac{\alpha - \beta_1}{\beta_2 - \beta_1}}, \tag{5.6c}$$

(3) *[energy and Morawetz estimate] for all $i' \in \{0, \dots, i\}$, $\alpha \in [\alpha_1, \alpha_2]$, and $t_2 \ge t_1 \ge t_0$,*

$$F(i', \alpha, t_2) + \int_{t_1}^{t_2} F(i' - 1, \alpha - 1, t) \mathrm{d}t \lesssim F(i', \alpha, t_1) + D t_1^{\alpha - \alpha_2 - \gamma}, \tag{5.6d}$$

and

(4) *[initial decay rate] if $\gamma > 0$, then for any $t \ge t_0$,*

$$F(i, \alpha_2, t) \lesssim t^{-\gamma} \left(F(i, \alpha_2, t_0) + D \right), \tag{5.6e}$$

then, for all $i' \in \{0, \dots, i\}$, all $\alpha \in [\max\{\alpha_1, \alpha_2 - i'\}, \alpha_2]$, and all $t \ge t_0$,

$$F(i - i', \alpha, t) \lesssim t^{\alpha - \alpha_2 - \gamma} (F(i, \alpha_2, t_0) + D), \tag{5.7}$$

where the implicit constant in $\lesssim$ can depend on α_2 and α_1.

Proof. Let $I = \lfloor \alpha_2 - \alpha_1 \rfloor \ge 1$. If $\gamma = 0$, then from the energy hypothesis (5.6d), one finds that the initial decay hypothesis estimate (5.6e) holds. Thus, in all cases, one finds for $t \ge t_0$,

$$F(i, \alpha_2, t) \lesssim t^{-\gamma} \left(F(i, \alpha_2, t_0) + D \right). \tag{5.8}$$

First, consider $\alpha_2 - \alpha \in \mathbb{N}$. For $i' \in \{1, \dots, I\}$ and $k \in \{0, 1\}$, observe that $F(i - i' + k, \alpha_2 - i' + k, t)$ satisfy

$$\begin{aligned} F(i - i' + k, \alpha_2 - i' + k, t_2) + \int_{t_1}^{t_2} F(i - i' + k - 1, \alpha_2 - i' + k - 1, t') \mathrm{d}t' \\ \lesssim F(i - i' + k, \alpha_2 - i' + k, t_1) + t_1^{-\gamma - i' + k} D. \end{aligned} \tag{5.9}$$

This combined with lemma 5.1 implies, for $t > 2t_0$,

$$F(i - i', \alpha_2 - i', t) \lesssim t^{-1} F(i - i' + 1, \alpha_2 - i' + 1, t/2) + t^{-1-\gamma-i'} D. \tag{5.10}$$

By induction, taking equation (5.8) as the base case and estimate (5.10) to justify the inductive step, we that, for all $i' \in \{0, \dots, I\}$ and $t \ge t_0$, there is the bound

$$F(i - i', \alpha_2 - i', t) \lesssim t^{-i' - \gamma} (F(i, \alpha_2, t_0) + D). \tag{5.11}$$

The same bound holds for $t \in [t_0, 2t_0]$ from (5.10) for $t \ge 2t_0$ and from the basic energy hypothesis (5.6d) and the monotonicity hypotheses (5.6a)-(5.6b).

Now, consider the case $\alpha \ge \alpha_2 - \lfloor \alpha_2 - \alpha_1 \rfloor$. Consider $i' \in \{0, \dots, I\}$ and $\zeta \in [0, i']$. From the interpolation hypothesis (5.6c) with $\alpha \to \alpha_2 - \zeta$, $\beta_1 \to \alpha_2 - i'$, $\beta_2 \to \alpha$, we get that for all $t \ge t_0$,

$$\begin{aligned} F(i - i', \alpha_2 - \zeta, t) &\lesssim F(i - i', \alpha_2 - i', t)^{\frac{\zeta}{i'}} F(i - i', \alpha_2, t)^{\frac{i' - \zeta}{i'}} \\ &\lesssim t^{-\zeta - \gamma} (F(i, \alpha_2, t_0) + D)^{\frac{\zeta}{i'}} F(i, \alpha_2, t)^{\frac{i' - \zeta}{i'}} \\ &\lesssim t^{-\zeta - \gamma} (F(i, \alpha_2, t_0) + D). \end{aligned} \tag{5.12}$$

Making the substitution $k = i - i'$ and $\alpha = \alpha_2 - \zeta \ge \alpha_2 - i' \ge \alpha_2 + k - i$ and using the monotonicity hypothesis (5.6b), one finds, for $k \in \{i - I, \dots, i\}$, $\alpha \in [\alpha_2 + k - i, \alpha_2]$, and $t \ge t_0$, there is the bound

$$F(k, \alpha, t) \lesssim t^{\alpha - \alpha_2 - \gamma} (F(i, \alpha_2, t_0) + D). \tag{5.13}$$

This gives the desired estimate for the cases $k \ge i - I$.

Finally, consider $\alpha < \alpha_2 - \lfloor \alpha_2 - \alpha_1 \rfloor$. Since $\alpha_2 - \alpha_1 < i$, one finds $I < i$, and since $\alpha_1 + 1 \geq \alpha_2 + (i - I) - i$, we have from the conclusion of the previous paragraph

$$F(i - I, \alpha_1 + 1, t) \lesssim t^{\alpha_1+1-\alpha_2-\gamma}(F(i, \alpha_2, t_0) + D). \tag{5.14}$$

Combining this with the energy and Morawetz hypothesis (5.6d) and lemma 5.1, one finds

$$F(i - I - 1, \alpha_1, t) \lesssim t^{\alpha_1-\alpha_2-\gamma}(F(i, \alpha_2, t_0) + D). \tag{5.15}$$

Interpolation now gives for all $\alpha \in [\alpha_1, \alpha_1 + 1]$ and $t \geq t_0$

$$F(i - I - 1, \alpha, t) \lesssim t^{\alpha-\alpha_2-\gamma}(F(i, \alpha_2, t_0) + D). \tag{5.16}$$

This combined with (5.13) implies for all $i' \in \{0, \dots, I + 1\}$, all $\alpha \in [\max\{\alpha_1, \alpha_2 - i'\}, \alpha_2]$, and all $t \geq t_0$,

$$F(i - i', \alpha, t) \lesssim t^{\alpha-\alpha_2-\gamma}(F(i, \alpha_2, t_0) + D). \tag{5.17}$$

The other monotonicity hypothesis (5.6a) then gives the desired estimate in the remaining cases. □

The following lemma states the $W^{i'}_\alpha$ norms squared satisfy the monotonicity and interpolation conditions for $f(i', \alpha, t)$ in lemma 5.2.

Lemma 5.3. *Let $i', i'_1, i'_2 \in \mathbb{N}$ and $\alpha, \beta, \beta_1, \beta_2 \in \mathbb{R}$. Let φ be a spin-weighted scalar. Let $t, t_1, t_2 \in [t_0, \infty)$.*

(1) [monotonicity] If $i'_1 \leq i'_2$ and $\beta_1 \leq \beta_2$, then

$$\|\varphi\|^2_{W^{i'_1}_\beta(\Sigma_t)} \lesssim \|\varphi\|^2_{W^{i'_2}_\beta(\Sigma_t)}, \tag{5.18a}$$

$$\|\varphi\|^2_{W^{i'}_{\beta_1}(\Sigma_t)} \lesssim \|\varphi\|^2_{W^{i'}_{\beta_2}(\Sigma_t)}. \tag{5.18b}$$

(2) [interpolation] If $\beta_1 \leq \alpha \leq \beta_2$, then

$$\|\varphi\|_{W^{i'}_\alpha(\Sigma_t)} \lesssim \|\varphi\|^{\frac{\alpha-\beta_1}{\beta_2-\beta_1}}_{W^{i'}_{\beta_1}(\Sigma_t)} \|\varphi\|^{\frac{\beta_2-\alpha}{\beta_2-\beta_1}}_{W^{i'}_{\beta_2}(\Sigma_t)}. \tag{5.19}$$

(3) [relation of spatial and spacetime norms]

$$\|\varphi\|^2_{W^{i'}_\beta(\Omega_{t_1,t_2})} = M^{-1} \int_{t_1}^{t_2} \|\varphi\|^2_{W^{i'}_\beta(\Sigma_t)} \mathrm{d}t. \tag{5.20}$$

Proof. The first monotonicity result follows from summing fewer non-negative terms. The second monotonicity result follows from the fact that $\beta_1 \leq \beta_2$ implies $r^{\beta_1} \lesssim r^{\beta_2}$. The interpolation result follows from Hölder's inequality. The relation between the spatial and spacetime norms follows from the definition of $\mathrm{d}^3\mu$ and $\mathrm{d}^4\mu$. □

5.2. **Spin-weighted transport equations.** Now we state a general lemma which provides energy and Morawetz estimates for the ingoing transport equation with source term satisfying energy and Morawetz estimates.

Lemma 5.4 (Y estimate)**.** *Let $\gamma \in (0, \infty)$ and $k \in \mathbb{N}$. Let $b_0(r)$ be a non-negative, smooth function defined in $\mathcal{M}$ such that $b_0(r) = MO_\infty(r^{-1})$.*

If φ and ϱ are scalars with spin weight s and φ satisfies

$$MY\varphi + b_0(r)\varphi = \varrho, \tag{5.21}$$

then for all $t_2 > t_1 \geq t_0$,

$$\|\varphi\|^2_{W^k_\gamma(\Sigma_{t_2})} + \|\varphi\|^2_{W^k_{\gamma-1}(\Omega_{t_1,t_2})} \lesssim_s \|\varphi\|^2_{W^k_\gamma(\Sigma_{t_1})} + \|\varrho\|^2_{W^k_{\gamma+1}(\Omega_{t_1,t_2})}, \tag{5.22a}$$

$$\|\varphi\|^2_{W^k_\gamma(\Sigma^{\text{int}}_{t_2})} + \|\varphi\|^2_{W^k_{\gamma-1}(\Omega^{\text{int}}_{t_1,t_2})} \lesssim_s \|\varphi\|^2_{W^k_\gamma(\Sigma^{\text{int}}_{t_1})} + \|\varrho\|^2_{W^k_{\gamma+1}(\Omega^{\text{int}}_{t_1,t_2})} + \|\varphi\|^2_{W^k_\gamma(\Xi_{t_1,t_2})}, \tag{5.22b}$$

$$\|\varphi\|^2_{W^k_\gamma(\Xi_{t_1,t_2})} + \|\varphi\|^2_{W^k_\gamma(\Sigma^{\text{ext}}_{t_2})} + \|\varphi\|^2_{W^k_{\gamma-1}(\Omega^{\text{ext}}_{t_1,t_2})} \lesssim_s \|\varphi\|^2_{W^k_\gamma(\Sigma^{\text{ext}}_{t_1})} + \|\varrho\|^2_{W^k_{\gamma+1}(\Omega^{\text{ext}}_{t_1,t_2})}, \tag{5.22c}$$

$$\|\varphi\|^2_{W^k_\gamma(\Sigma^{\text{ext}}_{t_0})} + \|\varphi\|^2_{W^k_{\gamma-1}(\Omega^{\text{early}}_{\text{init},t_0})} \lesssim_s \|\varphi\|^2_{W^k_\gamma(\Sigma_{\text{init}})} + \|\varrho\|^2_{W^k_{\gamma+1}(\Omega^{\text{early}}_{\text{init},t_0})}, \tag{5.22d}$$

and, for $t \geq t_0 + h(t_0)$ as in definition 2.29,

$$\|\varphi\|^2_{W^k_\gamma(\mathcal{H}^+_{t,\infty})} + \|\varphi\|^2_{W^k_\gamma(\Sigma^{\text{int}}_t)} + \|\varphi\|^2_{W^k_{\gamma-1}(\Omega^{\text{near}}_{t,\infty})} \lesssim_s \|\varphi\|^2_{W^k_\gamma(\Xi_{t_{\mathcal{C}}(t),\infty})} + \|\varrho\|^2_{W^k_{\gamma+1}(\Omega^{\text{near}}_{t,\infty})}. \tag{5.23}$$

The implicit constants in the above estimates depend on only γ and k.

Proof. Consider the case $k = 0$ first. Multiplying (5.21) by $M^{-1}r^\gamma\bar{\varphi}$, taking the real part, and applying the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned} Y(\tfrac{1}{2}r^\gamma|\varphi|^2) + (\tfrac{\gamma}{2} + M^{-1}rb_0(r))r^{\gamma-1}|\varphi|^2 &= r^\gamma\Re\{M^{-1}\varrho\bar{\varphi}\} \\ &\leq \tfrac{\gamma}{4}r^{\gamma-1}|\varphi|^2 + \gamma^{-1}M^{-2}r^{\gamma+1}|\varrho|^2. \end{aligned} \tag{5.24}$$

Absorbing the $|\varphi|^2$ on the right in to the left and multiplying with $M^{-\gamma-1}$ gives

$$Y(\tfrac{1}{2}M^{-\gamma-1}r^\gamma|\varphi|^2) + (\tfrac{\gamma}{4} + M^{-1}rb_0(r))M^{-(\gamma-1)-2}r^{\gamma-1}|\varphi|^2 \leq \gamma^{-1}M^{-(\gamma+1)-2}r^{\gamma+1}|\varrho|^2. \tag{5.25}$$

The energy and Morawetz estimate (5.22a) for $k = 0$ then follows from integrating over Ω_{t_1,t_2} with the measure $d^4\mu$ and the fact that $dt_aY^a = h'(r)$ is uniformly equivalent to 1. Here, we have dropped the positive flux at future null infinity. In an analogous way, the $k = 0$ case of the remaining estimates in (5.22) follows by integrating (5.25) with the measure $d^4\mu$ over $\Omega^{\text{int}}_{t_1,t_2}$, $\Omega^{\text{ext}}_{t_1,t_2}$, and $\Omega^{\text{early}}_{\text{init},t_0}$, respectively, and the $k = 0$ case of (5.23) follows from integrating with the measure $d^4\mu$ over $\Omega^{\text{near}}_{t,\infty}$ and $\Omega^{\text{near}}_{t,\infty} \cap \{t' \leq t\}$ such that the first integration gives the first and third term on the left of (5.23) and the second integration gives the second term on the left. Here, we made use of the facts that $(dt_a - dr_a)Y^a = 1 + h'(r)$ and $dt_aY^a = h'(r)$ are both uniformly equivalent to 1 and $dv_aY^a = 0$.

Now assume the result holds for some $k \geq 0$. From the fact that the operators $M\mathcal{L}_\xi$, $\mathring{\eth}$, $\mathring{\eth}'$ commute with Y, it follows that the estimates (5.22) and (5.23) hold for k but with (φ, ϱ) replaced by these derivatives operated on (φ, ϱ). Hence, the estimates (5.22) and (5.23) hold for k but with (φ, ϱ) replaced by any of $\{(\varphi, \varrho), (M\mathcal{L}_\xi\varphi, M\mathcal{L}_\xi\varrho), (\mathring{\eth}\varphi, \mathring{\eth}\varrho), (\mathring{\eth}'\varphi, \mathring{\eth}'\varrho)\}$.

If we commute (5.21) with $V(r\cdot)$, then, because of the first relation in (2.41), we have

$$\begin{aligned} &MYV(r\varphi) + (\tfrac{M}{r} + b_0(r))V(r\varphi) \\ &= V(r\varrho) + \tfrac{Mr(r^2-a^2)}{(r^2+a^2)^2}\varrho + \Big(\tfrac{\Delta}{2(r^2+a^2)}(M - r^2\partial_r(b_0(r))) - \tfrac{rM(r^2-a^2)}{(r^2+a^2)^2}(M + rb_0(r))\Big)\tfrac{\varphi}{r} + \tfrac{2aMr^2\mathcal{L}_\eta\varphi}{(r^2+a^2)^2} \\ &= V(r\varrho) + \tfrac{Mr(r^2-a^2)}{(r^2+a^2)^2}\varrho + MO_\infty(r^{-1})\varphi + M^2O_\infty(r^{-2})\mathcal{L}_\eta\varphi. \end{aligned} \tag{5.26}$$

This equation is in the form of equation of (5.21), so it remains to control the $W^k_{\gamma+1}(\Omega)$ norm squared of the right-hand side of (5.26), Ω being the region Ω_{t_1,t_2}, $\Omega^{\text{int}}_{t_1,t_2}$, $\Omega^{\text{ext}}_{t_1,t_2}$, $\Omega^{\text{early}}_{\text{init},t_0}$, or $\Omega^{\text{near}}_{t,\infty}$, which one integrates over. The $W^k_{\gamma+1}(\Omega)$ norm squared of the first two terms is clearly bounded by the $W^{k+1}_{\gamma+1}(\Omega)$ norm squared of ϱ itself. The last two terms are bounded by

$$\begin{aligned} &\|Mr^{-1}\varphi\|^2_{W^k_{\gamma+1}(\Omega)} + \|M^2r^{-2}\mathcal{L}_\eta\varphi\|^2_{W^k_{\gamma+1}(\Omega)} \\ &\lesssim \|\varphi\|^2_{W^k_{\gamma-1}(\Omega)} + \|\mathcal{L}_\eta\varphi\|^2_{W^k_{\gamma-3}(\Omega)} \\ &\lesssim_s \|\varphi\|^2_{W^k_{\gamma-1}(\Omega)} + \|\mathring{\eth}\varphi\|^2_{W^k_{\gamma-1}(\Omega)} + \|\mathring{\eth}'\varphi\|^2_{W^k_{\gamma-1}(\Omega)}. \end{aligned} \tag{5.27}$$

Therefore, the right-hand side is bounded by

$$\|f\|^2_{W^{k+1}_{\gamma+1}(\Omega)} + \|\varphi\|^2_{W^k_{\gamma-1}(\Omega)} + \|\mathring{\eth}\varphi\|^2_{W^k_{\gamma-1}(\Omega)} + \|\mathring{\eth}'\varphi\|^2_{W^k_{\gamma-1}(\Omega)}. \tag{5.28}$$

We have estimates for the last four terms from the previous paragraph, and by adding those estimates, the desired estimates (5.22) and (5.23) hold for $k + 1$. By induction, the estimates (5.22) and (5.23) hold for all $k \in \mathbb{N}$. ☐

5.3. **Spin-weighted wave equations.** The following is a standard r^p argument following the ideas originally given in [16]. Essentially one uses the vector-field method with the vector $M(1 + M^\delta r^{-\delta})Y + M^{-\alpha+1}r^\alpha V$ with $\delta > 0$ small and $\alpha \in [\delta, 2 - \delta]$. Since we use p for a spinoral weight, we use α for the exponent traditionally denoted by p in the r^p argument.

Lemma 5.5 (r^p estimates for spin-weighted waves in weighted energy spaces)**.** *Let $\delta > 0$ be sufficiently small. Let*[8] $|s| \leq 3$*. Let b_V, b_ϕ, and b_0 be real, smooth functions of r such that*

(1) $\exists b_{V,-1} \in \mathbb{R}$ such that $b_V = b_{V,-1}r + MO_\infty(1)$ and $b_{V,-1} \geq 0$,
(2) $b_\phi = MO_\infty(r^{-1})$, and
(3) $\exists b_{0,0} \in \mathbb{R}$ such that $b_0 = b_{0,0} + MO_\infty(r^{-1})$ and $b_{0,0} + |s| + s \geq 0$.

Let χ_1 be decreasing, smooth, 1 on $(-\infty, 0)$, and 0 on $(1,\infty)$, and let $\chi = \chi_1((R_0 - r)/M)$.[9]

Given these, there are constants $\bar{R}_0 = \bar{R}_0(b_0, b_\phi, b_V)$ and $C = C(b_0, b_\phi, b_V)$ such that for all scalars φ and ϑ with spin weight s, and if

$$\widehat{\boxed{S}}_s\varphi + b_V V\varphi + b_\phi \mathcal{L}_\eta\varphi + b_0\varphi = \vartheta, \tag{5.29}$$

then for all $R_0 \geq \bar{R}_0$, $t_2 \geq t_1 < t_0$, and $\alpha \in [\delta, 2-\delta]$,

$$\begin{aligned}
&\|rV\varphi\|^2_{W^0_{\alpha-2}(\Sigma^{R_0}_{t_2})} + \|\varphi\|^2_{W^1_{-2}(\Sigma^{R_0}_{t_2})} \\
&+ \|\varphi\|^2_{W^1_{\alpha-3}(\Omega^{R_0}_{t_1,t_2})} + \|MY\varphi\|^2_{W^0_{-1-\delta}(\Omega^{R_0}_{t_1,t_2})} \\
&+ \|\varphi\|^2_{F^0(\mathscr{I}^+_{t_1,t_2})} \\
&\leq C\Bigg(\|rV\varphi\|^2_{W^0_{\alpha-2}(\Sigma^{R_0}_{t_1})} + \|\varphi\|^2_{W^1_{-2}(\Sigma^{R_0}_{t_1})} \\
&\qquad + \|\varphi\|^2_{W^1_0(\Omega^{R_0-M,R_0}_{t_1,t_2})} + \sum_{t\in\{t_1,t_2\}} \|\varphi\|^2_{W^1_\alpha(\Sigma^{R_0-M,R_0}_t)} + \|\vartheta\|^2_{W^0_{\alpha-3}(\Omega^{R_0-M}_{t_1,t_2})} \Bigg).
\end{aligned} \tag{5.30}$$

Proof. The proof uses the method of multipliers with a multiplier that is a cut-off version of $M(1 + M^\delta r^{-\delta})Y + M^{1-\alpha}r^\alpha V$ with $\alpha \in [2\delta, 2-2\delta]$ and a rescaling $\delta \mapsto \delta/2$ will be made at the end of the proof. Within this proof, the relation $\lesssim$ is used to denote $\lesssim_{b_0,b_\phi,b_V,\bar{R}_0}$, and we use mass normalization as in definition 4.4.

Because the conformally regular functions are dense in the W^k_α spaces, by applying a density argument, it is sufficient to assume that ϑ and the initial data for φ are conformally regular. In particular, it is sufficient to assume that φ is conformally regular. This simplifies the treatment of certain terms on $\mathscr{I}^+$.

Step 1: Set up the method of multipliers. From equations (2.36e) and (2.39a), the spin-weighted wave equation (5.29) can be expanded out as

$$\begin{aligned}
&\left(2(r^2+a^2)YV + b_V V + (b_\phi + c_\phi)\mathcal{L}_\eta + (b_0 + c_0)\right)\varphi \\
&+ \left(-2\,\mathring{\eth}\,\mathring{\eth}' - f_1(\theta)\mathcal{L}_\xi\mathcal{L}_\xi - f_2(\theta)\mathcal{L}_\eta\mathcal{L}_\xi - f_3(\theta)\mathcal{L}_\xi\right)\varphi - \vartheta = 0,
\end{aligned} \tag{5.31}$$

where

$$c_\phi = -\frac{2ar}{a^2+r^2}, \qquad c_0 = \frac{(a^4 - 4Ma^2r + a^2r^2 + 2Mr^3)}{(a^2+r^2)^2}, \tag{5.32a}$$

$$f_1(\theta) = a^2\sin^2\theta, \qquad f_2(\theta) = 2a, \qquad f_3(\theta) = -2ias\cos\theta. \tag{5.32b}$$

Observe that, for each s, the f_i are smooth functions on the sphere such that $\mathcal{L}_\eta f_i = 0$. Thus, the spin-weighted wave equation (5.29) can be rewritten as

$$\sum_{i=1}^9 I_i = 0, \tag{5.33}$$

where

$$\begin{aligned}
&I_1 = 2(r^2+a^2)YV\varphi, && I_2 = b_V V\varphi, && I_3 = (b_\phi + c_\phi)\mathcal{L}_\eta\varphi, && I_4 = (b_0 + c_0)\varphi, \\
&I_5 = -2\,\mathring{\eth}\,\mathring{\eth}'\varphi, && I_6 = -f_1(\theta)\mathcal{L}_\xi\mathcal{L}_\xi\varphi, && I_7 = -f_2(\theta)\mathcal{L}_\eta\mathcal{L}_\xi\varphi, && I_8 = -f_3(\theta)\mathcal{L}_\xi\varphi, \\
&I_9 = -\vartheta.
\end{aligned} \tag{5.34}$$

Following the standard method-of-multipliers procedure, one can multiply the spin-weighted wave equation (5.33) by $\chi^2M^{1-\alpha}r^\alpha(V\bar{\varphi}) + \chi^2M(1+M^\delta r^{-\delta})(Y\bar{\varphi})$, multiply by a further factor of

[8] The range of s is essentially arbitrary, but a larger range of s requires redefining t with larger values of C_{hyp}.

[9] This implies that χ vanishes for $r \leq R_0 - M$ and is identically 1 for $r \geq R_0$.

$M^2/(r^2+a^2)$, take the real part, integrate the resulting equation over $\Omega^{R_0-M}_{t_1,t_2}$, and then estimate the various terms. To do so, it is convenient to introduce, for $i \in \{1,\dots,9\}$,

$$I_{i,V} = \Re\left(\chi^2 M^{1-\alpha} r^\alpha (V\bar\varphi)\frac{M^2}{r^2+a^2} I_i\right), \quad I_{i,Y} = \Re\left(\chi^2 M(1+M^\delta r^{-\delta})(Y\bar\varphi)\frac{M^2}{r^2+a^2} I_i\right). \tag{5.35}$$

For $i \in \{1,\dots,9\}$ and $X \in \{V,Y\}$, the term $I_{i,X}$ is said to be put in standard form when there are $P_{i,X}$, $\Pi_{i,X,\text{principal}}$, and $\Pi_{i,X,\text{error}}$ such that, for any region $\Omega = \Omega_{t,r} \times S^2$ with $\Omega_{t,r} \subset \mathbb{R}\times(2M,\infty)$ and with boundary $\partial\Omega$,

$$\int_\Omega I_{i,X}\mathrm{d}^4\mu = \int_{\partial\Omega} \nu_a P^a_{i,X}\mathrm{d}^3\mu_\nu + \int_\Omega (\Pi_{i,X,\text{principal}} + \Pi_{i,X,\text{error}})\mathrm{d}^4\mu. \tag{5.36}$$

After the method of multipliers presented in the first step of this proof, the purpose of step 2 is to isolate the principal terms, both in the bulk $\Omega^{R_0-M}_{t_1,t_2}$ and in energies on the $\Sigma^{R_0-M}_t$. The $I_{1,V}$ and $I_{2,V}$ contribute the dominant $|V\varphi|^2$ terms both on the $\Sigma^{R_0-M}_t$ and in $\Omega^{R_0-M}_{t_1,t_2}$, the $I_{1,Y}$ term contributes the dominant $|Y\varphi|$ term on the $\Sigma^{R_0-M}_t$ and in $\Omega^{R_0-M}_{t_1,t_2}$, the $I_{5,Y}$ term contributes the dominant $|\mathring{\eth}'\varphi|^2$ term on $\Sigma^{R_0-M}_t$, but the $I_{4,V}$ and $I_{5,V}$ terms together contribute the dominant $|\varphi|^2$ and $|\mathring{\eth}'\varphi|^2$ term in $\Omega^{R_0-M}_{t_1,t_2}$. Step 3 is to define the remaining, nonprincipal terms.

The I_6 and I_7 terms are particularly difficult to treat. The $I_{6,Y}$ and $I_{7,Y}$ contribute terms that do not decay in r faster than those that arise in the principal terms. To handle these, it is necessary to exploit the largeness of C_{hyp}, which is set in definition 4.1. Step 4 treats the principal part in $\Omega^{R_0-M}_{t_1,t_2}$. Step 5 treats the energy on each Σ_t, and in particular the $I_{6,V}$ and $I_{7,V}$ terms. Step 6 treats the flux through $\mathscr{I}^+$. Step 7 treats the remaining bulk terms, which completes the proof.

The remainder of this proof uses mass normalization, as in definition 4.4.

Step 2: Definition of the principal terms. Within this proof, the principal terms are those that contribute a nonnegative, leading-order term, either in the bulk or on hypersurfaces. To isolate pure powers of r in the principal bulk terms, instead of powers of r^2+a^2, it is useful to observe

$$\frac{1}{r^2} - \frac{1}{r^2+a^2} = \frac{a^2}{r^2(r^2+a^2)}. \tag{5.37}$$

Integrating $I_{1,V} = \chi^2 r^\alpha (r^2+a^2)^{-1}\Re((V\bar\varphi)(2(r^2+a^2)YV\varphi))$ and applying Y integration-by-parts formula (4.37a), one finds $I_{1,V}$ is in standard form with

$$P^a_{1,V} = \chi^2 r^\alpha |V\varphi|^2 Y^a, \tag{5.38a}$$
$$\Pi_{1,V,\text{principal}} = \chi^2 r^{\alpha-1}\alpha|V\varphi|^2, \tag{5.38b}$$
$$\Pi_{1,V,\text{error}} = \partial_r(\chi^2) r^\alpha |V\varphi|^2. \tag{5.38c}$$

The term $I_{2,V} = \chi^2 r^\alpha (r^2+a^2)^{-1}\Re((V\bar\varphi)(b_V V\varphi))$ can immediately be put in standard form with

$$P^a_{2,V} = 0, \tag{5.39a}$$
$$\Pi_{2,V,\text{principal}} = \chi^2 r^{\alpha-1} b_{V,-1}|V\varphi|^2, \tag{5.39b}$$
$$\Pi_{2,V,\text{error}} = \chi^2 r^\alpha\left(\frac{-b_{V,-1} r a^2}{r^2(r^2+a^2)} + \frac{b_V - r b_{V,-1}}{r^2+a^2}\right)|V\varphi|^2. \tag{5.39c}$$

Integrating $I_{1,Y}$ and applying commutator formula (2.41), one finds

$$
\begin{aligned}
\int_{\Omega^{R_0-M}_{t_1,t_2}} I_{1,Y}\mathrm{d}^4\mu ={}& \int_{\Omega^{R_0-M}_{t_1,t_2}} 2\chi^2(1+r^{-\delta})\Re\left((Y\bar\varphi)(YV\varphi)\right)\mathrm{d}^4\mu \\
={}& \int_{\Omega^{R_0-M}_{t_1,t_2}} 2\chi^2(1+r^{-\delta})\Re\left((Y\bar\varphi)(VY\varphi)\right)\mathrm{d}^4\mu \\
&+ \int_{\Omega^{R_0-M}_{t_1,t_2}} 2\chi^2(1+r^{-\delta})(Y\bar\varphi)\frac{r^2-a^2}{(r^2+a^2)^2}(Y\varphi)\mathrm{d}^4\mu \\
&+ \int_{\Omega^{R_0-M}_{t_1,t_2}} 2\chi^2(1+r^{-\delta})\Re\left((Y\bar\varphi)\frac{2ar}{(r^2+a^2)^2}(\mathcal{L}_\eta\varphi)\right)\mathrm{d}^4\mu.
\end{aligned}
\tag{5.40}
$$

Now, applying V integration-by-parts formula (4.37b) to the first term on the right, one finds $I_{1,Y}$ in standard form

$$
\begin{aligned}
P^a_{1,Y} &= \chi^2(1+r^{-\delta})|Y\varphi|^2V^a, && (5.41a)\\
\Pi_{1,Y,\text{principal}} &= \frac{1}{2}\delta\chi^2r^{-\delta-1}|Y\varphi|^2, && (5.41b)\\
\Pi_{1,Y,\text{error}} &= \Pi_{1,Y,(Y,Y)} + \Pi_{1,Y,(Y,\eta)}, && (5.41c)\\
\Pi_{1,Y,(Y,Y)} &= \Bigg(-\frac{1}{2}\delta\chi^2r^{-\delta-1} - \partial_r\left(\chi^2r^{-\delta}\frac{\Delta}{2(r^2+a^2)}\right) \\
&\qquad + 2\chi^2(1+r^{-\delta})\frac{r^2-a^2}{(r^2+a^2)^2}\Bigg)|Y\varphi|^2, && (5.41d)\\
\Pi_{1,Y,(Y,\eta)} &= 2\chi^2(1+r^{-\delta})\frac{2ar}{(r^2+a^2)^2}\Re\left((Y\bar\varphi)(\mathcal{L}_\eta\varphi)\right). && (5.41e)
\end{aligned}
$$

The term $I_{5,V}$ can be rewritten, using $\mathring{ð}\,\bar\varphi = \overline{\mathring{ð}'\,\varphi}$, as

$$
I_{5,V} = -2\Re\left(\mathring{ð}\left(\chi^2r^\alpha\frac{1}{r^2+a^2}(V\bar\varphi)(\mathring{ð}'\,\varphi)\right)\right) + 2\Re\left(\chi^2r^\alpha\frac{1}{r^2+a^2}(V\overline{\mathring{ð}'\,\varphi})(\mathring{ð}'\,\varphi)\right). \tag{5.42}
$$

Thus, applying the $\mathring{ð}$ and V integration-by-parts formulae (4.24) and (4.37b), one finds $I_{5,V}$ is in standard form with

$$
\begin{aligned}
P^a_{5,V} &= \chi^2r^\alpha\frac{1}{r^2+a^2}|\mathring{ð}'\,\varphi|^2V^a, && (5.43a)\\
\Pi_{5,V,\text{principal}} &= \frac{2-\alpha}{2}\chi^2r^{\alpha-3}|\mathring{ð}'\,\varphi|^2, && (5.43b)\\
\Pi_{5,V,\text{error}} &= \left(\frac{2-\alpha}{2}\chi^2r^{\alpha-3} - \partial_r\left(\chi^2r^\alpha\frac{\Delta}{2(r^2+a^2)^2}\right)\right)|\mathring{ð}'\,\varphi|^2. && (5.43c)
\end{aligned}
$$

Similarly, for the $I_{5,Y}$ term,

$$
I_{5,Y} = -2\Re\left(\mathring{ð}\left(\chi^2\frac{1+r^{-\delta}}{r^2+a^2}(Y\bar\varphi)(\mathring{ð}'\,\varphi)\right)\right) + 2\Re\left(\chi^2\frac{1+r^{-\delta}}{r^2+a^2}(Y\,\mathring{ð}\,\bar\varphi)(\mathring{ð}'\,\varphi)\right), \tag{5.44}
$$

so that Y integration-by-parts formula (4.37a) gives the standard form with

$$
\begin{aligned}
P^a_{5,Y} &= \chi^2\left(1+\frac{1}{r^\delta}\right)\frac{1}{r^2+a^2}|\mathring{ð}'\,\varphi|^2Y^a, && (5.45a)\\
\Pi_{5,Y,\text{principal}} &= 0, && (5.45b)\\
\Pi_{5,Y,\text{error}} &= \left(\partial_r\left(\chi^2(1+r^{-\delta})\frac{1}{r^2+a^2}\right)\right)|\mathring{ð}'\,\varphi|^2. && (5.45c)
\end{aligned}
$$

Integrating $I_{4,V} = \chi^2 r^\alpha (r^2+a^2)^{-1}\Re((V\bar\varphi)(b_0+c_0)\varphi)$ and applying V integration-by-parts formula (4.37b), one obtains the standard form with

$$\begin{aligned}
P^a_{4,V} &= \frac{1}{2}\chi^2\frac{r^\alpha}{r^2+a^2}(b_0+c_0)|\varphi|^2 V^a, && (5.46a)\\
\Pi_{4,V,\text{principal}} &= \frac{2-\alpha}{4}b_{0,0}\chi^2 r^{\alpha-3}|\varphi|^2, && (5.46b)\\
\Pi_{4,V,\text{error}} &= \left(-\frac{2-\alpha}{4}b_{0,0}\chi^2 r^{\alpha-3} + \partial_r\left(r^\alpha\frac{\Delta}{4(r^2+a^2)^2}(b_0+c_0)\right)\right)|\varphi|^2. && (5.46c)
\end{aligned}$$

For $(i,X)\in\{(2,Y),(3,V),(3,Y),(4,Y),(6,V),(6,Y),(7,V),(7,Y),(8,V),(8,Y),(9,V),(9,Y)\}$ -that is, for all (i,X) for which $\Pi_{i,X,\text{principal}}$ has not yet been defined- define

$$\Pi_{i,X,\text{principal}} = 0. \tag{5.47}$$

Step 3: Define the remaining terms. Considering $I_{6,Y}$ and isolating a total ξ derivative, one finds

$$\begin{aligned}
I_{6,Y} ={}& -\chi^2(r^2+a^2)^{-1}(1+r^{-\delta})\Re\left((Y\bar\varphi)f_1(\theta)\mathcal{L}_\xi\mathcal{L}_\xi\varphi\right)\\
={}& -\mathcal{L}_\xi\left(\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((Y\bar\varphi)f_1(\theta)\mathcal{L}_\xi\varphi\right)\right)\\
&+\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((Y\mathcal{L}_\xi\bar\varphi)f_1(\theta)\mathcal{L}_\xi\varphi\right).
\end{aligned}\tag{5.48}$$

Now integrating and applying Y integration-by-parts formula (4.37a), one obtains the standard form for $I_{6,Y}$ with

$$\begin{aligned}
P^a_{6,Y} ={}& -\chi^2\frac{1+r^{-\delta}}{r^2+a^2}f_1(\theta)\Re\left((Y\bar\varphi)\mathcal{L}_\xi\varphi\right)\xi^a\\
&+\frac{1}{2}\chi^2\frac{1+r^{-\delta}}{r^2+a^2}f_1(\theta)|\mathcal{L}_\xi\varphi|^2 Y^a, && (5.49a)\\
\Pi_{6,Y,\text{error}} ={}& \left(\partial_r\left(\frac{1}{2}\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\right)\right)|\mathcal{L}_\xi\varphi|^2 f_1(\theta). && (5.49b)
\end{aligned}$$

(Recall all the principal terms were defined in the previous step.)

The term $I_{7,Y}$ can be rewritten, using the Leibniz rule in $\mathcal{L}_\xi$, Y, and $\mathcal{L}_\eta$, as

$$\begin{aligned}
&-\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((Y\bar\varphi)f_2(\theta)\mathcal{L}_\eta\mathcal{L}_\xi\varphi\right)\\
={}& -\mathcal{L}_\xi\left(\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((Y\bar\varphi)f_2(\theta)\mathcal{L}_\eta\varphi\right)\right)+\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((\mathcal{L}_\xi Y\bar\varphi)f_2(\theta)\mathcal{L}_\eta\varphi\right)\\
={}& -\mathcal{L}_\xi\left(\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((Y\bar\varphi)f_2(\theta)\mathcal{L}_\eta\varphi\right)\right)+Y\left(\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((\mathcal{L}_\xi\bar\varphi)f_2(\theta)\mathcal{L}_\eta\varphi\right)\right)\\
&+\partial_r\left(\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\right)f_2(\theta)\Re\left((\mathcal{L}_\xi\bar\varphi)\mathcal{L}_\eta\varphi\right)-\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((\mathcal{L}_\xi\bar\varphi)f_2(\theta)Y\mathcal{L}_\eta\varphi\right)\\
={}& -\mathcal{L}_\xi\left(\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((Y\bar\varphi)f_2(\theta)\mathcal{L}_\eta\varphi\right)\right)+Y\left(\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((\mathcal{L}_\xi\bar\varphi)f_2(\theta)\mathcal{L}_\eta\varphi\right)\right)\\
&+\partial_r\left(\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\right)f_2(\theta)\Re\left((\mathcal{L}_\xi\bar\varphi)\mathcal{L}_\eta\varphi\right)-\mathcal{L}_\eta\left(\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((\mathcal{L}_\xi\bar\varphi)f_2(\theta)Y\varphi\right)\right)\\
&+\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((\mathcal{L}_\xi\mathcal{L}_\eta\bar\varphi)f_2(\theta)Y\varphi\right).
\end{aligned}\tag{5.50}$$

Now, identifying the final term as the opposite of the term on the first line, one can integrate to obtain the standard form for $I_{7,Y}$ with

$$\begin{aligned} P^a_{7,Y} =& -\frac{1}{2}\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((Y\bar{\varphi})f_2(\theta)\mathcal{L}_\eta\varphi\right)\xi^a \\ &+\frac{1}{2}\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\Re\left((\mathcal{L}_\xi\bar{\varphi})f_2(\theta)\mathcal{L}_\eta\varphi\right)Y^a, \end{aligned} \tag{5.51a}$$

$$\Pi_{7,Y,\mathrm{error}} = \frac{1}{2}\partial_r\left(\chi^2\frac{1+r^{-\delta}}{r^2+a^2}\right)\Re\left((\mathcal{L}_\xi\bar{\varphi})f_2(\theta)\mathcal{L}_\eta\varphi\right). \tag{5.51b}$$

$I_{6,V}$ can be rewritten, by isolating a total ξ derivative, as

$$I_{6,V} = \mathcal{L}_\xi\left(-\chi^2 f_1(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((V\overline{\varphi})\mathcal{L}_\xi\varphi\right)\right) + \chi^2 f_1(\theta)\frac{r^\alpha}{r^2+a^2}\Re((V\overline{\mathcal{L}_\xi\varphi})(\mathcal{L}_\xi\varphi)). \tag{5.52}$$

Since $\mathcal{L}_\xi$ acting on a scalar and in the (t,r,ω) parametrization is just ∂_t, if one integrates the first term in t and applies V integration-by-parts formula (4.37b) on the second, then one obtains $I_{6,V}$ in standard form with

$$P^a_{6,V} = -\chi^2 f_1(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((V\overline{\varphi})\mathcal{L}_\xi\varphi\right)\xi^a + \frac{1}{2}\chi^2 f_1(\theta)\frac{r^\alpha}{r^2+a^2}|\mathcal{L}_\xi\varphi|^2 V^a, \tag{5.53a}$$

$$\Pi_{6,V,\mathrm{error}} = -\frac{1}{4}\partial_r\left(\chi^2 f_1(\theta)r^\alpha\frac{\Delta}{(r^2+a^2)^2}\right)|\mathcal{L}_\xi\varphi|^2. \tag{5.53b}$$

This type of analysis can be applied to $I_{7,V}$. Term $I_{7,V}$ can be rewritten using the Leibniz rule in $\mathcal{L}_\xi$, V, and $\mathcal{L}_\eta$ as

$$\begin{aligned} I_{7,V} =& -\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((V\overline{\varphi})\mathcal{L}_\eta\mathcal{L}_\xi\varphi\right) \\ =& \mathcal{L}_\xi\left(-\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((V\overline{\varphi})\mathcal{L}_\eta\varphi\right)\right) + \chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((\mathcal{L}_\xi V\overline{\varphi})\mathcal{L}_\eta\varphi\right) \\ =& \mathcal{L}_\xi\left(-\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((V\overline{\varphi})\mathcal{L}_\eta\varphi\right)\right) + V\left(\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((\mathcal{L}_\xi\overline{\varphi})\mathcal{L}_\eta\varphi\right)\right) \\ &- V\left(\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\right)\Re\left((\mathcal{L}_\xi\overline{\varphi})\mathcal{L}_\eta\varphi\right) - \chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((\mathcal{L}_\xi\overline{\varphi})\mathcal{L}_\eta V\varphi\right) \\ =& \mathcal{L}_\xi\left(-\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((V\overline{\varphi})\mathcal{L}_\eta\varphi\right)\right) + V\left(\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((\mathcal{L}_\xi\overline{\varphi})\mathcal{L}_\eta\varphi\right)\right) \\ &- V\left(\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\right)\Re\left((\mathcal{L}_\xi\overline{\varphi})\mathcal{L}_\eta\varphi\right) - \mathcal{L}_\eta\left(\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((\mathcal{L}_\xi\overline{\varphi})V\varphi\right)\right) \\ &+ \chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((\mathcal{L}_\eta\mathcal{L}_\xi\overline{\varphi})V\varphi\right). \end{aligned} \tag{5.54}$$

Identifying the last term on the right with the opposite of the term on the left, one obtains

$$\begin{aligned} 2I_{7,V} =& -2\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((V\overline{\varphi})\mathcal{L}_\eta\mathcal{L}_\xi\varphi\right) \\ =& \mathcal{L}_\xi\left(-\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((V\overline{\varphi})\mathcal{L}_\eta\varphi\right)\right) + V\left(\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((\mathcal{L}_\xi\overline{\varphi})\mathcal{L}_\eta\varphi\right)\right) \\ &- \mathcal{L}_\eta\left(\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((\mathcal{L}_\xi\overline{\varphi})V\varphi\right)\right) - V\left(\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\right)\Re\left((\mathcal{L}_\xi\overline{\varphi})\mathcal{L}_\eta\varphi\right). \end{aligned} \tag{5.55}$$

Since $\mathcal{L}_\xi$ and $\mathcal{L}_\eta$ are ∂_t and ∂_ϕ in the (t,r,θ,ϕ) coordinates, from the V integration by parts formula (4.35b), one obtains the standard form for $I_{7,V}$ with

$$P^a_{7,V} = -\frac{1}{2}\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re((V\overline{\varphi})\mathcal{L}_\eta\varphi)\xi^a + \frac{1}{2}\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re((\mathcal{L}_\xi\overline{\varphi})\mathcal{L}_\eta\varphi)V^a, \tag{5.56a}$$

$$\Pi_{7,V,\mathrm{error}} = -\left(\partial_r\left(\frac{1}{2}\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\frac{\Delta}{2(r^2+a^2)}\right)\right)\Re((\mathcal{L}_\xi\overline{\varphi})\mathcal{L}_\eta\varphi). \tag{5.56b}$$

For $(i, X) \in \{(2, Y), (3, V), (3, Y), (4, Y), (8, V), (8, Y), (9, V), (9, Y)\}$ -that is, for all (i, X) for which $P^a_{i,X}$ has not yet been defined- define

$$P^a_{i,X} = 0. \tag{5.57}$$

Step 4: Treat the principal bulk term. Let

$$\begin{aligned}\Pi_{\text{principal}} &= \sum_{i\in\{1,\ldots,9\},X\in\{V,Y\}} \Pi_{i,X,\text{principal}}\\ &= \Pi_{1,V,\text{principal}} + \Pi_{2,V,\text{principal}} + \Pi_{1,Y,\text{principal}} + \Pi_{5,V,\text{principal}} + \Pi_{4,V,\text{principal}}\\ &= (\alpha + b_{V,-1})\chi^2 r^{\alpha-1}|V\varphi|^2 + \frac{1}{2}\delta\chi^2 r^{-\delta-1}|Y\varphi|^2\\ &\quad + \frac{2-\alpha}{2}\chi^2 r^{\alpha-3}|\mathring{\eth}'\varphi|^2 + \frac{2-\alpha}{4} b_{0,0}\chi^2 r^{\alpha-3}|\varphi|^2.\end{aligned} \tag{5.58}$$

Since $b_{V,-1} \geq 0$, by assumption, that term can be dropped. It is convenient to rewrite $|\mathring{\eth}'\varphi|^2$ as $\left(|\mathring{\eth}'\varphi|^2 - \frac{|s|+s}{2}|\varphi|^2\right) + \frac{|s|+s}{2}|\varphi|^2$ and to observe that, when integrated over spheres, both these summands are nonnegative from the lower bound on $\mathring{\eth}'$ in lemma 4.25. Thus,

$$\begin{aligned}\int_{\Omega^{R_0-M}_{t_1,t_2}} \Pi_{\text{principal}} \mathrm{d}^4\mu \geq \int_{\Omega^{R_0-M}_{t_1,t_2}} \chi^2 \Bigg(& \alpha r^{\alpha-1}|V\varphi|^2 + \frac{1}{2}\delta r^{-\delta-1}|Y\varphi|^2\\ &+ \frac{2-\alpha}{2} r^{\alpha-3}\left(|\mathring{\eth}'\varphi|^2 - \frac{|s|+s}{2}|\varphi|^2\right)\\ &+ \frac{2-\alpha}{4}\left(b_{0,0} + |s| + s\right) r^{\alpha-3}|\varphi|^2\Bigg)\mathrm{d}^4\mu.\end{aligned} \tag{5.59}$$

Thus, using the Hardy inequality (4.42), one finds, for some positive constants C_1, C_2, C_3, C_4,

$$\begin{aligned}&\int_{\Omega^{R_0-M}_{t_1,t_2}} \Pi_{\text{principal}} \mathrm{d}^4\mu + C_1\|\varphi\|^2_{W^1_0(\Omega^{R_0-M,R_0}_{t_1,t_2})}\\ &\quad\geq \int_{\Omega^{R_0-M}_{t_1,t_2}} \Bigg(C_2 r^{\alpha-1}|V\varphi|^2 + C_3 r^{-\delta-1}|Y\varphi|^2\\ &\qquad + \frac{2-\alpha}{2} r^{\alpha-3}\left(|\mathring{\eth}'\varphi|^2 - \frac{|s|+s}{2}|\varphi|^2\right)\\ &\qquad + \frac{2-\alpha}{4}\left(b_{0,0} + |s| + s + C_4\right) r^{\alpha-3}|\varphi|^2\Bigg)\mathrm{d}^4\mu.\end{aligned} \tag{5.60}$$

Since the hypothesis of the theorem assumes that $b_{0,0} + |s| + s \geq 0$ and $C_4 > 0$, all the coefficients are strictly positive. Furthermore, the terms that they multiply are all nonnegative. Given that there are positive multiples of $|\mathring{\eth}'\varphi|^2 - \frac{|s|+s}{2}|\varphi|^2$ and $|\varphi|^2$, both with coefficients of $r^{\alpha-3}$, these can be lower bounded by positive multiples of $|\mathring{\eth}'\varphi|^2$ and $|\varphi|^2$. Hence, there are constants C_1, C_2 such that

$$\begin{aligned}&\int_{\Omega^{R_0-M}_{t_1,t_2}} \Pi_{\text{principal}} \mathrm{d}^4\mu + C_1\|\varphi\|^2_{W^1_0(\Omega^{R_0-M,R_0}_{t_1,t_2})}\\ &\geq C_2 \int_{\Omega^{R_0-M}_{t_1,t_2}} \left(r^{\alpha-1}|V\varphi|^2 + r^{-\delta-1}|Y\varphi|^2 + r^{\alpha-3}|\mathring{\eth}'\varphi|^2 + r^{\alpha-3}|\varphi|^2\right)\mathrm{d}^4\mu.\end{aligned} \tag{5.61}$$

The relation (2.35) gives $\mathcal{L}_\xi = V + \xi^Y Y + \xi^\eta \mathcal{L}_\eta$ with coefficients satisfying the bounds $|\xi^Y| \lesssim 1$, and $|\xi^\eta| \lesssim Mr^{-2}$, from which it follows that

$$\int_{\Omega^{R_0-M}_{t_1,t_2}} r^{-1-\delta}|\mathcal{L}_\xi\varphi|^2 \mathrm{d}^4\mu \lesssim \int_{\Omega_{t_1,t_2}} \Pi_{\text{principal}} \mathrm{d}^4\mu + \|\varphi\|^2_{W^1_0(\Omega^{R_0-M,R_0}_{t_1,t_2})}. \tag{5.62}$$

Step 5: Treat the energy on hyperboloids. On hyperboloids, the energies can be decomposed into the principal and error terms. Unfortunately, the error energies for the $I_{6,Y}$ and $I_{7,Y}$ terms have coefficients that are of the same order in r as those in the principal part. Fortunately, we can use C_{hyp} as a large parameter to dominate these error terms by the principal parts. All the remaining terms are strictly lower order and, hence, easily dominated.

First, define the principal terms. To do so, it is also useful to recall, cf. (2.47d), (2.45), that

$$\lim_{r\to\infty} \mathrm{d}t_a Y^a = 2, \tag{5.63a}$$

$$\lim_{r\to\infty} \frac{r^2}{M^2}\mathrm{d}t_a V^a = C_{\text{hyp}}. \tag{5.63b}$$

On the hyperboloids, let

$$e_{1,V,\text{principal}} = 2\chi^2 r^\alpha |V\varphi|^2, \tag{5.64a}$$

$$e_{1,Y,\text{principal}} = C_{\text{hyp}}\chi^2 r^{-2}|Y\varphi|^2, \tag{5.64b}$$

$$e_{5,Y,\text{principal}} = 2\chi^2 r^{-2}|\mathring{\eth}'\varphi|^2. \tag{5.64c}$$

For $(i,X) \in \{(2,V), (2,Y), (3,V), (3,Y), (4,V), (4,Y), (5,V), (6,V), (6,Y), (7,V), (7,Y), (8,V), (8,Y), (9,V), (9,Y)\}$ -that is for all (i,X) for which $e_{i,X,\text{principal}}$ has not been defined- define

$$e_{i,X,\text{principal}} = 0. \tag{5.65}$$

Define

$$\begin{aligned} e_{\text{principal}} &= \sum_{i\in\{1,\dots,9\},X\in\{V,Y\}} e_{i,X} \\ &= e_{1,V,\text{principal}} + e_{1,Y,\text{principal}} + e_{5,Y,\text{principal}} \\ &= 2\chi^2 r^\alpha|V\varphi|^2 + C_{\text{hyp}}\chi^2 r^{-2}|Y\varphi|^2 + 2\chi^2 r^{-2}|\mathring{\eth}'\varphi|^2. \end{aligned} \tag{5.66}$$

There are some useful lower bounds to observe. First, note that since r^α can be taken to be larger than any given constant by taking r sufficiently large, and since $C_{\text{hyp}} \geq \frac{1}{2}$, it follows from the Hardy lemma 4.31 that for any sufficiently small ε if R_0 is sufficiently large, then

$$\begin{aligned} \int_{\Sigma_t^{R_0-M}} \chi^2 r^{-2}|\varphi|^2 \mathrm{d}^3\mu &\leq (16+\varepsilon)\int_{\Sigma_t^{R_0-M}} \chi^2|V\varphi|^2\mathrm{d}^3\mu + \varepsilon\int_{\Sigma_t^{R_0-M}} \chi^2 r^{-2}|Y\varphi|^2\mathrm{d}^3\mu \\ &\quad + \int_{\Sigma_t^{R_0-M,R_0}} |\varphi|^2\mathrm{d}^3\mu \\ &\leq \int_{\Sigma_t^{R_0-M}} e_{\text{principal}}\mathrm{d}^3\mu + \int_{\Sigma_t^{R_0-M,R_0}} |\varphi|^2\mathrm{d}^3\mu. \end{aligned} \tag{5.67}$$

Lemma 4.26 controls the integral of $|\mathcal{L}_\eta\varphi|^2$ on the sphere. Integrating in r and then applying equation (5.67), one finds

$$\begin{aligned} \int_{\Sigma_t^{R_0-M}} \chi^2 r^{-2}|\mathcal{L}_\eta\varphi|^2\mathrm{d}^3\mu &\leq \int_{\Sigma_t^{R_0-M}} \left(2\chi^2 r^{-2}|\mathring{\eth}'\varphi|^2 + \chi^2 r^{-2}s^2|\varphi|^2\right)\mathrm{d}^3\mu \\ &\leq (1+s^2)\int_{\Sigma_t^{R_0-M}} e_{\text{principal}}\mathrm{d}^3\mu + s^2\int_{\Sigma_t^{R_0-M,R_0}} |\varphi|^2\mathrm{d}^3\mu. \end{aligned} \tag{5.68}$$

Furthermore, due to equation (2.35) and since $|a| \leq M$, for R_0/M sufficiently large relative to C_{hyp} and s, one has, for $r \geq R_0$,

$$|\mathcal{L}_\xi\varphi|^2 \leq 3|V\varphi|^2 + \frac{3}{4}|Y\varphi|^2 + \frac{1}{C_{\text{hyp}}(1+s^2)}|\mathcal{L}_\eta\varphi|^2. \tag{5.69}$$

Multiplying by $\chi^2 r^{-2}$, integrating in r, using the bound (5.68) to control the final term, the definitions of $e_{1,V,\text{principal}}$ and $e_{1,Y,\text{principal}}$ to control the first two terms, using the largeness of r^α in $e_{1,V,\text{principal}}$, and the factor of C_{hyp} in $e_{1,Y,\text{principal}}$, one finds

$$\int_{\Sigma_t^{R_0-M}} \chi^2 r^{-2}|\mathcal{L}_\xi\varphi|^2\mathrm{d}^3\mu \leq \frac{4}{C_{\text{hyp}}}\int_{\Sigma_t^{R_0-M}} e_{\text{principal}}\mathrm{d}^3\mu + \frac{1}{C_{\text{hyp}}}\int_{\Sigma_t^{R_0-M,R_0}} |\varphi|^2\mathrm{d}^3\mu. \tag{5.70}$$

Let

$$e_{1,V,\mathrm{error}} = (-2 + (\mathrm{d}t_a Y^a))\chi^2 r^\alpha |V\varphi|^2, \tag{5.71a}$$
$$e_{1,Y,\mathrm{error}} = \chi^2 \left((-C_{\mathrm{hyp}} r^{-2} + (\mathrm{d}t_a V^a)) + (\mathrm{d}t_a V^a) r^{-\delta}\right) |Y\varphi|^2, \tag{5.71b}$$
$$e_{4,V,\mathrm{error}} = (\mathrm{d}t_a V^a)\chi^2 r^\alpha \frac{1}{2}\frac{1}{r^2+a^2}(b_0 + c_0)|\varphi|^2, \tag{5.71c}$$
$$e_{5,V,\mathrm{error}} = (\mathrm{d}t_a V^a)\chi^2 r^\alpha \frac{1}{r^2+a^2} |\mathring{\eth}'\varphi|^2, \tag{5.71d}$$
$$e_{5,Y,\mathrm{error}} = \chi^2 \left(-\frac{2}{r^2} + \frac{\mathrm{d}t_a Y^a}{r^2+a^2} + (\mathrm{d}t_a Y^a)\frac{1}{r^\delta}\frac{1}{r^2+a^2}\right) |\mathring{\eth}'\varphi|^2, \tag{5.71e}$$
$$e_{6,V,\mathrm{error}} = e_{6,V,(V\xi)} + e_{6,V,(\xi\xi)}, \tag{5.71f}$$
$$e_{6,V,(V\xi)} = -(\mathrm{d}t_a \xi^a)\chi^2 f_1(\theta)\frac{r^\alpha}{r^2+a^2}\Re\left((V\overline{\varphi})\mathcal{L}_\xi\varphi\right), \tag{5.71g}$$
$$e_{6,V,(\xi\xi)} = \frac{1}{2}(\mathrm{d}t_a V^a)\chi^2 f_1(\theta)\frac{r^\alpha}{r^2+a^2}|\mathcal{L}_\xi\varphi|^2, \tag{5.71h}$$
$$e_{7,V,\mathrm{error}} = e_{7,V,(V\eta)} + e_{7,V,(\xi\eta)}, \tag{5.71i}$$
$$e_{7,V,(V\eta)} = -(\mathrm{d}t_a \xi^a)\frac{1}{2}\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re((V\overline{\varphi})\mathcal{L}_\eta\varphi), \tag{5.71j}$$
$$e_{7,V,(\xi\eta)} = (\mathrm{d}t_a V^a)\frac{1}{2}\chi^2 f_2(\theta)\frac{r^\alpha}{r^2+a^2}\Re((\mathcal{L}_\xi\overline{\varphi})\mathcal{L}_\eta\varphi), \tag{5.71k}$$

and, turning to the terms that are harder to estimate, let

$$e_{6,Y,\mathrm{error}} = e_{6,Y,(Y\xi)} + e_{6,Y,(\xi\xi)}, \tag{5.72a}$$
$$e_{6,Y,(Y\xi)} = -(\mathrm{d}t_a \xi^a)\chi^2 \frac{1+r^{-\delta}}{r^2+a^2} f_1(\theta)\Re((Y\bar{\varphi})\mathcal{L}_\xi\varphi), \tag{5.72b}$$
$$e_{6,Y,(\xi\xi)} = (\mathrm{d}t_a Y^a)\frac{1}{2}\chi^2 \frac{1+r^{-\delta}}{r^2+a^2} f_1(\theta)|\mathcal{L}_\xi\varphi|^2, \tag{5.72c}$$
$$e_{7,Y,\mathrm{error}} = e_{7,Y,(Y\eta)} + e_{7,Y,(\xi\eta)}, \tag{5.72d}$$
$$e_{7,Y,(Y\eta)} = -(\mathrm{d}t_a \xi^a)\frac{1}{2}\chi^2 \frac{1+r^{-\delta}}{r^2+a^2} f_2(\theta)\Re\left((Y\bar{\varphi})\mathcal{L}_\eta\varphi\right), \tag{5.72e}$$
$$e_{7,Y,(\xi\eta)} = (\mathrm{d}t_a Y^a)\frac{1}{2}\chi^2 \frac{1+r^{-\delta}}{r^2+a^2} f_2(\theta)\Re\left((\mathcal{L}_\xi\bar{\varphi})\mathcal{L}_\eta\varphi\right). \tag{5.72f}$$

For $(i, X) \in \{(2, V), (2, Y), (3, V), (3, Y), (4, Y), (6, V), (6, Y), (7, V), (7, Y), (8, V), (8, Y), (9, V), (9, Y)\}$ -that is for all (i, X) for which $e_{i,X,\mathrm{error}}$ has not been defined- let

$$e_{i,X,\mathrm{error}} = 0. \tag{5.73}$$

For r sufficiently large, one has $|\mathrm{d}t_a Y^a| \leq 4$, $|\mathrm{d}t_a \xi^a| \leq 2$, and $1 + r^{-\delta} \leq 2$. Independently of r, one has $|f_1(\theta)| \leq M^2$ and $|f_2(\theta)| \leq 2M$. Thus, from the previous bounds

$$\begin{aligned}
&\left|\int_{\Sigma_t^{R_0-M}} \left(e_{6,Y,(Y\xi)} + e_{6,Y,(\xi\xi)} + e_{7,Y,(Y\eta)} + e_{7,Y,(\xi\eta)}\right) \mathrm{d}^3\mu\right| \\
&\quad\leq \int_{\Sigma_t^{R_0-M}} \left(4r^{-2}|\mathcal{L}_\xi\varphi||Y\varphi| + 4r^{-2}|\mathcal{L}_\xi\varphi|^2\right) \mathrm{d}^3\mu \\
&\qquad + \int_{\Sigma_t^{R_0-M}} \left(4r^{-2}|Y\varphi||\mathcal{L}_\eta\varphi| + 8r^{-2}|\mathcal{L}_\xi\varphi||\mathcal{L}_\eta\varphi|\right) \mathrm{d}^3\mu.
\end{aligned} \tag{5.74}$$

Every one of these terms has a factor of either $|\mathcal{L}_\xi\varphi|$ or $|Y\varphi|$, so that one obtains a factor of $C_{\mathrm{hyp}}^{-1/2}$ either from the coefficient of $|Y\varphi|^2$ in the definition of $e_{1,Y,\mathrm{principal}}$ or from the bound on $|\mathcal{L}_\xi\varphi|^2$ in inequality (5.70). Thus, from the Cauchy-Schwarz inequality, from introducing a factor of $C_{\mathrm{hyp}}^{-1/2}$ on the $\mathcal{L}_\eta$ derivatives when applying the Cauchy-Schwarz inequality, and from equations

(5.64b), (5.68), (5.70), one finds

$$\begin{aligned}
&\left| \int_{\Sigma_t^{R_0-M}} \left(e_{6,Y,(Y\xi)} + e_{6,Y,(\xi\xi)} + e_{7,Y,(Y\eta)} + e_{7,Y,(\xi\eta)}\right) \mathrm{d}^3\mu \right| \\
&\quad \leq \int_{\Sigma_t^{R_0-M}} \left(\left(2 + 2C_{\mathrm{hyp}}^{1/2}\right) |Y\varphi|^2 + \left(2 + 4 + 4C_{\mathrm{hyp}}^{1/2}\right) |\mathcal{L}_\xi\varphi|^2 \right) r^{-2} \mathrm{d}^3\mu \\
&\qquad + \int_{\Sigma_t^{R_0-M}} \left(\frac{2}{C_{\mathrm{hyp}}^{1/2}} + \frac{4}{C_{\mathrm{hyp}}^{1/2}} \right) |\mathcal{L}_\eta\varphi|^2 r^{-2} \mathrm{d}^3\mu \\
&\quad \leq \left(\frac{2 + 2C_{\mathrm{hyp}}^{1/2}}{C_{\mathrm{hyp}}} + \frac{4(6 + 4C_{\mathrm{hyp}}^{1/2})}{C_{\mathrm{hyp}}} + \frac{6(1+s^2)}{C_{\mathrm{hyp}}^{1/2}} \right) \int_{\Sigma_t^{R_0-M}} e_{\mathrm{principal}} \mathrm{d}^3\mu \\
&\qquad + \left(\frac{6 + 4C_{\mathrm{hyp}}^{1/2}}{C_{\mathrm{hyp}}} + \frac{6s^2}{C_{\mathrm{hyp}}^{1/2}} \right) \int_{\Sigma_t^{R_0-M,R_0}} |\varphi|^2 \mathrm{d}^3\mu.
\end{aligned} \tag{5.75}$$

Since s^2 is bounded by 9, and since C_{hyp} is chosen to be 10^6 in definition 4.1, it follows that, for some constant C, on any hyperboloid there is the bound

$$\begin{aligned}
&\left| \int_{\Sigma_t^{R_0-M}} \left(e_{6,Y,(Y\xi)} + e_{6,Y,(\xi\xi)} + e_{7,Y,(Y\eta)} + e_{7,Y,(\xi\eta)}\right) \mathrm{d}^3\mu \right| \\
&\quad \leq \frac{1}{2} \int_{\Sigma_t^{R_0-M}} e_{\mathrm{principal}} \mathrm{d}^3\mu + C \int_{\Sigma_t^{R_0-M,R_0}} |\varphi|^2 \mathrm{d}^3\mu.
\end{aligned} \tag{5.76}$$

It can now be shown that the remaining error terms can be made arbitrarily small relative to $e_{\mathrm{principal}}$ by taking r sufficiently large. One way to show this is to show that the term consists of a norm squared appearing in $e_{\mathrm{principal}}$ but with a lower exponent. For example, in $e_{1,V,\mathrm{error}}$, there is a factor of $|V\varphi|^2$ with an exponent that vanishes at a rate of $r^{\alpha-1}$ (since $-2 + \mathrm{d}t_a Y^a$ vanishes as r^{-1}), which decays faster than the r^α coefficient of $|V\varphi|^2$ in $e_{\mathrm{principal}}$. Another, similar, method is to show that the term involves the (real part of) the inner product of two terms involving φ, each of which appear in $e_{\mathrm{principal}}$, and that the coefficient of this inner product vanishes faster the geometric mean of the corresponding coefficients for the terms in $e_{\mathrm{principal}}$. For example, the term $e_{6,V,(V\xi)}$ has a factor of $\Re((V\overline{\varphi})\mathcal{L}_\xi\varphi)$ multiplied by a coefficient that vanishes as $r^{\alpha-2}$. The geometric mean of two terms that decay with a particular exponent decays with an exponent that is given by the arithmetic mean. The energy $e_{\mathrm{principal}}$, dominates $r^\alpha|\varphi|^2$ and $r^{-2}|\mathcal{L}_\xi\varphi|^2$, and the exponents satisfy $\alpha - 2 < ((\alpha) + (-2))/2$, so, by taking r sufficiently large, one can ensure that $e_{6,V,(V\xi)}$ is arbitrarily small relative to $e_{\mathrm{principal}}$. Thus, for all the error terms, it is simply a matter of checking the relevant exponents, which are given in the following table.

Term	Exponent	Exponent from $e_{\mathrm{principal}}$
$(1, V)$	$\alpha - 1$	α
$(1, Y)$	$-\delta - 2$	-2
$(4, V)$	$(-2) + (\alpha - 2)$	-2
$(5, V)$	$(-2) + (\alpha - 2)$	-2
$(5, Y)$	$-\delta - 2$	-2
$(6, V, (V\xi))$	$\alpha - 2$	$((\alpha) + (-2))/2$
$(6, V, (\xi\xi))$	$(-2) + (\alpha - 2)$	-2
$(7, V, (V\eta))$	$\alpha - 2$	$((\alpha) + (-2))/2$
$(7, V, (\xi\eta))$	$(-2) + (\alpha - 2)$	-2

On the level sets of t, one has that ν can be chosen to be $\mathrm{d}t$. Furthermore, one has $\mathrm{d}^3\mu_\nu = \mathrm{d}^3\mu$. From this and the definitions in the previous paragraph, one finds for all (i, X) that

$$\int_{\Sigma_t^{R_0-M}} \nu_a P^a_{i,X} \mathrm{d}^3\mu_\nu = \int_{\Sigma_t^{R_0-M}} \left(e_{i,X,\mathrm{principal}} + e_{i,X,\mathrm{error}}\right) \mathrm{d}^3\mu. \tag{5.77}$$

Thus, one can conclude

$$\int_{\Sigma_{t_2}^{R_0}} e_{\text{principal}} \mathrm{d}^3\mu \lesssim \int_{\Sigma_{t_2}^{R_0-M}} \sum_{i\in\{1,\dots,9\},X\in\{V,Y\}} \nu_a P^a_{i,X} \mathrm{d}^3\mu_\nu + \|\varphi\|^2_{W^1_0(\Sigma_{t_2}^{R_0-M,R_0})}, \tag{5.78a}$$

$$\int_{\Sigma_{t_1}^{R_0-M}} \sum_{i\in\{1,\dots,9\},X\in\{V,Y\}} \nu_a P^a_{i,X} \mathrm{d}^3\mu_\nu \lesssim \int_{\Sigma_{t_1}^{R_0}} e_{\text{principal}} \mathrm{d}^3\mu + \|\varphi\|^2_{W^1_0(\Sigma_{t_1}^{R_0-M,R_0})}. \tag{5.78b}$$

Step 6: Treat the flux through $\mathscr{I}^+_{t_1,t_2}$. In this step, it is useful to treat $\mathscr{I}^+_{t_1,t_2}$ as the limit as $r\to\infty$ of a sequence of surfaces given in hyperboloidal coordinates by $[t_1,t_2]\times\{r\}\times S^2$ but to think of this in the conformal geometry.

The only nonvanishing $P^a_{i,X}$ arise from $(i,X)\in\{(1,V),\ (1,Y),\ (4,V),\ (5,V),\ (5,Y),\ (6,V),\ (6,Y),\ (7,V),\ (7,Y)\}$. The normal to the surfaces of constant r is $\nu=\mathrm{d}r$, so $\nu_aY^a\sim -1$, $\nu_aV^a\sim 1$, and $\nu_a\xi^a=0$. From conformal regularity, one finds that $r^{\alpha-2}|\varphi|^2_k\to 0$. Thus,

$$\begin{aligned}0 &= \int_{\mathscr{I}^+_{t_1,t_2}} \nu_a P^a_{1,V}\mathrm{d}^3\mu_\nu = \int_{\mathscr{I}^+_{t_1,t_2}} \nu_a P^a_{5,V}\mathrm{d}^3\mu_\nu = \int_{\mathscr{I}^+_{t_1,t_2}} \nu_a P^a_{5,Y}\mathrm{d}^3\mu_\nu = \int_{\mathscr{I}^+_{t_1,t_2}} \nu_a P^a_{4,V}\mathrm{d}^3\mu_\nu \\ &= \int_{\mathscr{I}^+_{t_1,t_2}} \nu_a P^a_{6,V}\mathrm{d}^3\mu_\nu = \int_{\mathscr{I}^+_{t_1,t_2}} \nu_a P^a_{7,V}\mathrm{d}^3\mu_\nu = \int_{\mathscr{I}^+_{t_1,t_2}} \nu_a P^a_{6,Y}\mathrm{d}^3\mu_\nu = \int_{\mathscr{I}^+_{t_1,t_2}} \nu_a P^a_{7,Y}\mathrm{d}^3\mu_\nu,\end{aligned} \tag{5.79}$$

and the only nonvanishing term is

$$\int_{\mathscr{I}^+_{t_1,t_2}} \nu_a P^a_{1,Y}\mathrm{d}^3\mu_\nu = \int_{\mathscr{I}^+_{t_1,t_2}} |Y\varphi|^2\mathrm{d}^3\mu_{\mathscr{I}} \geq 0. \tag{5.80}$$

Step 7: Treat the remaining terms in the bulk via the Cauchy-Schwarz inequality. The same type of analysis as in step 5 can be used to show that the bulk error terms are all small relative to $\int_{\Omega_{t_1,t_2}^{R_0-M}} \Pi_{\text{principal}}\mathrm{d}^4\mu + \|\varphi\|^2_{W^1_0(\Omega_{t_1,t_2}^{R_0-M,R_0})}$. The following table shows that the exponents satisfy the relevant bound, with $-\infty$ standing in when the error term decays faster than polynomially or is compactly supported. Note that many of the relevant exponents arise from the cancellation of leading-order terms. Note also that $(1,Y,(Y,Y))$ and $(1,Y,(Y,\eta))$ are used to denote $\Pi_{1,Y,(Y,Y)}$ and $\Pi_{1,Y,(Y,\eta)}$ respectively.

Term	Exponent	Exponent from $e_{\text{principal}}$
$(1,V)$	$-\infty$	$\alpha-1$
$(2,V)$	$\alpha-2$	$\alpha-1$
$(1,Y,(Y,Y))$	-2	$-\delta-1$
$(1,Y,(Y,\eta))$	-3	$((-\delta-1)+(\alpha-3))/2$
$(5,V)$	$\alpha-4$	$\alpha-3$
$(5,Y)$	-3	$\alpha-3$
$(4,V)$	$\alpha-4$	$\alpha-3$
$(6,Y)$	-3	$-\delta-1$
$(7,Y)$	-3	$-\delta-1$
$(6,V)$	$\alpha-3$	$-\delta-1$
$(7,V)$	$\alpha-3$	$((-\delta-1)+(\alpha-3))/2$
$(3,V)$	$\alpha-3$	$((\alpha-1)+(\alpha-3))/2$
$(8,V)$	$\alpha-2$	$((\alpha-1)+(-\delta-1))/2$
$(2,Y)$	-1	$((-\delta-1)+(\alpha-1))/2$
$(3,Y)$	-3	$((-\delta-1)+(\alpha-3))/2$
$(8,Y)$	-2	$((-\delta-1)+(-\delta-1))/2$

For the $(2,Y)$ term to be controlled, it is necessary that $(-\delta+\alpha-2)/2>1$, which is why the proof has so far considered $\alpha>2\delta$.

It remains to treat the I_9 terms. For any $\varepsilon>0$,

$$|I_{9,V}| \lesssim \varepsilon\chi^2 r^{\alpha-1}|V\varphi|^2 + \varepsilon^{-1}\chi^2 r^{\alpha-3}|\vartheta|, \tag{5.81a}$$

$$|I_{9,Y}| \lesssim \varepsilon\chi^2 r^{-1-\delta}|Y\varphi|^2 + \varepsilon^{-1}\chi^2 r^{\delta-3}|\vartheta|^2. \tag{5.81b}$$

For ε sufficiently small, the first term on the right of each of these bounds is dominated by $\Pi_{\text{principal}}$. Thus from the fact that all the error terms can be made small relative to the principal terms (plus some additional term for $r \in [R_0 - M, R_0]$), one finds

$$\begin{aligned}
&\int_{\Sigma_{t_2}^{R_0-M}} \sum_{i\in\{1,\dots,9\},X\in\{V,Y\}} \nu_a P^a_{i,X} \mathrm{d}^3\mu_\nu + \int_{\Omega^{R_0}_{t_1,t_2}} \Pi_{\text{principal}} \mathrm{d}^4\mu + \int_{\mathscr{I}^+_{t_1,t_2}} |Y\varphi|^2 \mathrm{d}^3\mu_{\mathscr{I}} \\
&\quad\lesssim \int_{\Sigma_{t_1}^{R_0-M}} \sum_{i\in\{1,\dots,9\},X\in\{V,Y\}} \nu_a P^a_{i,X} \mathrm{d}^3\mu_\nu + \|\varphi\|^2_{W^1_0(\Omega^{R_0-M,R_0}_{t_1,t_2})} \\
&\quad\quad + \int_{\Omega^{R_0-M}_{t_1,t_2}} r^{\alpha-3}|\vartheta| \mathrm{d}^3\mu.
\end{aligned} \tag{5.82}$$

From this, from, the estimates (5.78b) and (5.78a) and, from the fact that we can add an extra term $\|\mathring{\eth}\varphi\|^2_{W^0_{\alpha-3}(\Omega^{R_0}_{t_1,t_2})}$ to the left because of the relation (4.26), it now follows that

$$\begin{aligned}
&\|rV\varphi\|^2_{W^0_{\alpha-2}(\Sigma^{R_0}_{t_2})} + \|\varphi\|^2_{W^1_{-2}(\Sigma^{R_0}_{t_2})} \\
&+ \|\varphi\|^2_{W^1_{\alpha-3}(\Omega^{R_0}_{t_1,t_2})} + \|MY\varphi\|^2_{W^0_{-1-\delta}(\Omega^{R_0}_{t_1,t_2})} \\
&+ \|\varphi\|^2_{F^0(\mathscr{I}^+_{t_1,t_2})} \\
&\leq C\bigg(\|rV\varphi\|^2_{W^0_{\alpha-2}(\Sigma^{R_0}_{t_1})} + \|\varphi\|^2_{W^1_{-2}(\Sigma^{R_0}_{t_1})} \\
&\qquad + \|\varphi\|^2_{W^1_0(\Omega^{R_0-M,R_0}_{t_1,t_2})} + \sum_{t\in\{t_1,t_2\}} \|\varphi\|^2_{W^1_\alpha(\Sigma^{R_0-M,R_0}_t)} + \|\vartheta\|^2_{W^0_{\alpha-3}(\Omega^{R_0-M}_{t_1,t_2})} \bigg).
\end{aligned} \tag{5.83}$$

The term $\|MY\varphi\|^2_{W^0_{-1-\delta}(\Omega^{R_0}_{t_1,t_2})}$ can trivially be replaced by $\|MY\varphi\|^2_{W^0_{-1-2\delta}(\Omega^{R_0}_{t_1,t_2})}$. Doing so and making the rescaling $\delta \mapsto \delta/2$, one obtains the desired result (5.30) for all $\alpha \in [\delta, 2-\delta]$. ☐

5.4. **Spin-weighted wave equations in higher regularity.** This section proves the analogue of the r^p-estimate for spin-weighted wave equations, from lemma 5.5, but in higher regularity.

Lemma 5.6 (Higher-regularity r^p-estimates for waves in weighted energy spaces)**.** *Under the same assumptions as in lemma 5.5 except that we now assume φ has spin weight $|s| \leq 2$, for any $k \in \mathbb{N}$, there are constants $\bar{R}_0 = \bar{R}_0(b_0, b_\phi, b_V)$ and $C = C(b_0, b_\phi, b_V)$ such that for all spin-weight s scalars φ and ϑ, and if (5.29) is satisfied, then for all $R_0 \geq \bar{R}_0$, $t_2 \leq t_1 \leq t_0$ and $\alpha \in [\delta, 2-\delta]$, there is*

$$\begin{aligned}
&\|rV\varphi\|^2_{W^k_{\alpha-2}(\Sigma^{R_0}_{t_2})} + \|\varphi\|^2_{W^{k+1}_{-2}(\Sigma^{R_0}_{t_2})} \\
&+ \|\varphi\|^2_{W^{k+1}_{\alpha-3}(\Omega^{R_0}_{t_1,t_2})} + \|MY\varphi\|^2_{W^k_{-1-\delta}(\Omega^{R_0}_{t_1,t_2})} \\
&+ \|\varphi\|^2_{F^k(\mathscr{I}^+_{t_1,t_2})} \\
&\leq C\bigg(\|rV\varphi\|^2_{W^k_{\alpha-2}(\Sigma^{R_0}_{t_1})} + \|\varphi\|^2_{W^{k+1}_{-2}(\Sigma^{R_0}_{t_1})} \\
&\qquad + \|\varphi\|^2_{W^{k+1}_0(\Omega^{R_0-M,R_0}_{t_1,t_2})} + \sum_{t\in\{t_1,t_2\}} \|\varphi\|^2_{W^{k+1}_\alpha(\Sigma^{R_0-M,R_0}_t)} + \|\vartheta\|^2_{W^k_{\alpha-3}(\Omega^{R_0-M}_{t_1,t_2})} \bigg).
\end{aligned} \tag{5.84}$$

Proof. For a given set of operators $\mathbb{X}$, consider the estimate

$$\begin{aligned}
\sum_{|\mathbf{a}|\leq k} \bigg(&\|\mathbb{X}^{\mathbf{a}} rV\varphi\|^2_{W^0_{\alpha-2}(\Sigma^{R_0}_{t_2})} + \|\mathbb{X}^{\mathbf{a}}\varphi\|^2_{W^1_{-2}(\Sigma^{R_0}_{t_2})} \\
&+ \|\mathbb{X}^{\mathbf{a}}\varphi\|^2_{W^1_{\alpha-3}(\Omega^{R_0}_{t_1,t_2})} + \|\mathbb{X}^{\mathbf{a}} MY\varphi\|^2_{W^0_{-1-\delta}(\Omega^{R_0}_{t_1,t_2})} \\
&+ \int_{\mathscr{I}^+_{t_1,t_2}} M|\mathbb{X}^{\mathbf{a}}\mathcal{L}_\xi\varphi|^2 \mathrm{d}^3\mu_{\mathscr{I}} \bigg)
\end{aligned}$$

$$\leq C \sum_{|\mathbf{a}|\leq k} \Big(\|rV\mathbb{X}^{\mathbf{a}}\varphi\|^2_{W^0_{\alpha-2}(\Sigma^{R_0}_{t_1})} + \|\mathbb{X}^{\mathbf{a}}\varphi\|^2_{W^1_{-2}(\Sigma^{R_0}_{t_1})} + \|\mathbb{X}^{\mathbf{a}}\varphi\|^2_{W^1_0(\Omega^{R_0-M,R_0}_{t_1,t_2})} + \sum_{t\in\{t_1,t_2\}} \|\mathbb{X}^{\mathbf{a}}\varphi\|^2_{W^1_\alpha(\Sigma^{R_0-M,R_0}_t)} + \|\mathbb{X}^{\mathbf{a}}\vartheta\|^2_{W^0_{\alpha-3}(\Omega^{R_0-M}_{t_1,t_2})}\Big). \tag{5.85}$$

In the following steps, the bound (5.85) is proved for an increasingly large sequence of sets of operators until the estimate is proved for $\mathbb{X}=\mathbb{D}$, which completes the proof.

Step 1: $\mathbb{X}=\{M\mathcal{L}_\xi\}$. Since $M\mathcal{L}_\xi$ commutes through the spin-weighted wave equation (5.29), any number of compositions of $M\mathcal{L}_\xi$ can be applied, and the original r^p bound (5.30) will hold with φ and ϑ replaced by $(M\mathcal{L}_\xi)^i\varphi$ and $(M\mathcal{L}_\xi)^i\vartheta$, which proves the higher-regularity r^p bound (5.85) with $\mathbb{X}=\{M\mathcal{L}_\xi\}$.

Step 2: $\mathbb{X}=\not{\mathbb{D}}$ with at most one angular derivative. If the spin weight is negative, $s<0$, then, commuting the original spin-weighted wave equation in its expanded form (5.31) with $\mathring{\eth}'$ and using the commutation relation (2.42d), one finds

$$\begin{aligned}\Big(2(r^2+a^2)YV + b_{V,\mathring{\eth}'}V + (b_{\phi,\mathring{\eth}'}+c_\phi)\mathcal{L}_\eta + (b_{0,\mathring{\eth}'}+c_0)\Big)\mathring{\eth}'\varphi \\ + \Big(-2\mathring{\eth}\mathring{\eth}' - f_1(\theta)\mathcal{L}_\xi\mathcal{L}_\xi - f_2(\theta)\mathcal{L}_\eta\mathcal{L}_\xi - f_3(\theta)\mathcal{L}_\xi\Big)\mathring{\eth}'\varphi - \vartheta_{\mathring{\eth}'} = 0,\end{aligned} \tag{5.86}$$

where

$$\begin{aligned} b_{V,\mathring{\eth}'} = b_V, \quad b_{\phi,\mathring{\eth}'} = b_\phi, \quad b_{0,\mathring{\eth}'} = b_0 - 2(s-1), \\ \vartheta_{\mathring{\eth}'} = \mathring{\eth}'\vartheta - \frac{1}{\sqrt{2}}(\mathring{\eth}' f_1(\theta))\mathcal{L}_\xi\mathcal{L}_\xi\varphi - \frac{1}{\sqrt{2}}(\mathring{\eth}' f_3(\theta))\mathcal{L}_\xi\varphi,\end{aligned} \tag{5.87}$$

and c_ϕ, c_0, and the f_i are given in equation (5.32). While in the case of $s\geq 0$, one can commute (5.31) with $\mathring{\eth}$ and apply the commutation relation (2.42d) to find that $\mathring{\eth}\varphi$ satisfies an equation of the form (5.86) with $b_{V,\mathring{\eth}'}$, $b_{\phi,\mathring{\eth}'}$, b_0 and $\vartheta_{\mathring{\eth}'}$ replaced by

$$\begin{aligned} b_{V,\mathring{\eth}} = b_V, \quad b_{\phi,\mathring{\eth}} = b_\phi, \quad b_{0,\mathring{\eth}} = b_0 + 2s, \\ \vartheta_{\mathring{\eth}} = \mathring{\eth}\vartheta - \frac{1}{\sqrt{2}}(\mathring{\eth} f_1(\theta))\mathcal{L}_\xi\mathcal{L}_\xi\varphi - \frac{1}{\sqrt{2}}(\mathring{\eth} f_3(\theta))\mathcal{L}_\xi\varphi,\end{aligned} \tag{5.88}$$

respectively.

These are in the form of the spin-weighted wave equation (5.29) from the r^p lemma 5.5. It is clear that if the first two assumptions in lemma 5.5, on the asymptotics of b_V and b_ϕ, held for the original wave equation, then they hold for $b_{V,\mathring{\eth}'}$ and $b_{\phi,\mathring{\eth}'}$ or for $b_{V,\mathring{\eth}}$ and $b_{\phi,\mathring{\eth}}$ respectively. The scalars $\mathring{\eth}'\varphi$ and $\mathring{\eth}\varphi$ have spin weight $s-1$ and $s+1$ respectively, and their spin weights lie in $\{-3,\cdots,3\}$. Furthermore, the leading-order parts of $b_{0,\mathring{\eth}'}$ and $b_{0,\mathring{\eth}}$ satisfy

$$\begin{aligned} b_{0,\mathring{\eth}',0} + |s-1| + (s-1) &= b_{0,0} - 2s + 2 = (b_{0,0}+|s|+s) + 2|s| + 2, && \text{for } s<0, \\ b_{0,\mathring{\eth},0} + |s+1| + (s+1) &= b_{0,0} + 4s + 2 = (b_{0,0}+|s|+s) + 2s + 2, && \text{for } s\geq 0,\end{aligned} \tag{5.89}$$

which means that if $b_{0,0}+|s|+s>0$, then $b_{0,\mathring{\eth}',0}+|s-1|+(s-1)>0$ and $b_{0,\mathring{\eth},0}+|s+1|+(s+1)>0$. In particular, if b_0 from the original equation (5.29) satisfies assumption (3) from the r^p lemma 5.5, then so do $b_{0,\mathring{\eth}'}$ from the commuted equation (5.86) and and $b_{0,\mathring{\eth}}$ from the analogue for $\mathring{\eth}\varphi$. Thus, if the original spin-weighted wave equation (5.29) satisfies the hypotheses of the r^p lemma 5.5, then so do the $\mathring{\eth}'$ or $\mathring{\eth}$ commuted equations.

Hence by applying the r^p lemma 5.5, the bound (5.30) holds if we replace φ and ϑ by $\mathring{\eth}'\varphi$ and the sum of $\mathring{\eth}'\vartheta$ and a $O_\infty(1)$ weight times at most two compositions of $M\mathcal{L}_\xi$ acting on φ. The terms involving compositions of $M\mathcal{L}_\xi$ acting on φ can be estimated by the higher-regularity r^p bound (5.85) with $\mathbb{X}=\{M\mathcal{L}_\xi\}$, which proves the higher regularity r^p bound (5.85) with $\mathbb{X}=\mathbb{S}$ in the special case where the multiindex $\mathbf{a}$ is restricted so that there is at most one angular derivative and it is either $\mathring{ð}'$ if $s<0$ and $\mathring{ð}$ if $s\geq 0$.

Step 3: $\mathbb{X}=\not{\mathbb{D}}$ without restriction on the number of angular derivatives. Since any $D\in\{M^2\mathcal{L}_\xi\mathcal{L}_\xi, M\mathcal{L}_\xi\mathcal{L}_\eta, \mathcal{L}_\eta\mathcal{L}_\eta, S_s\}$ commutes with the homogeneous part of the wave equation (5.29), the r^p estimate (5.85) follows trivially if we replace φ and ϑ by $D\varphi$ and $D\vartheta$, respectively. In view of the relation (2.39a) between S_s and $\mathring{S}_s$, the estimate (5.85) holds if $\mathbb{X}=\{M^2\mathcal{L}_\xi\mathcal{L}_\xi, M\mathcal{L}_\xi\mathcal{L}_\eta, \mathcal{L}_\eta\mathcal{L}_\eta, \mathring{S}_s\}$.

Consider now the higher-regularity r^p bound (5.85) with $\mathbb{X}=\not{\mathbb{D}}$. First, consider the case where there is a sum up to an even order $2i$ of angular derivatives. By lemma 4.24, the corresponding norms can be replaced by norms involving $\mathring{S}_s^i$, and such norms were already controlled in the previous paragraph. Now, consider the case where there is a sum up to an odd order $2i+1$ of angular derivatives. By the previous argument, all the terms of order up to $2i$ can be replaced by norms defined in terms of $\mathring{S}_s$. Since the lower-order terms are controlled, by equation (4.26) and the previous argument, the terms involving $2i+1$ derivatives can be controlled by terms involving lower-order terms and terms involving $\mathring{S}_s^i$ and either $\mathring{ð}'$ or $\mathring{ð}$ depending on whether $s<0$ or $s\geq 0$. Such terms can be controlled by combining the arguments of the previous paragraph and step 2.

Note that in step 2 in equations (5.87)-(5.88), ϑ was replaced by the sum of one angular derivative acting on ϑ and a $O_\infty(1)$ coefficient of at most two compositions of $M\mathcal{L}_\xi$ acting on φ. Applying compositions of $M\mathcal{L}_\xi$, $\mathcal{L}_\eta$, or $\mathring{S}_s$ of total order $k-1$ to an angular derivative of ϑ will give terms bounded by $|\vartheta|^2_{k,\not{\mathbb{D}}}$. Similarly, applying compositions of $M\mathcal{L}_\xi$, $\mathcal{L}_\eta$, or $\mathring{S}_s$ of total order $k-1$ to at most two compositions of $M\mathcal{L}_\xi$ acting on φ will give terms bounded by $|\not{\mathbb{D}}^{\mathbf{a}}\varphi|^2$, in which either $|\mathbf{a}|\leq k-1$ or such that at least one term in $\not{\mathbb{D}}^{\mathbf{a}}$ is a $M\mathcal{L}_\xi$ derivative. In either case, by first proving the r^p bound (5.85) to order k with $\mathbb{X}=\{M\mathcal{L}_\xi\}$ and then proving the bound with $\mathbb{X}=\not{\mathbb{D}}$ with increasing orders $i\leq k$, one finds that all the terms arising of the form $|\not{\mathbb{D}}^{\mathbf{a}}\varphi|^2$ are controlled by earlier bounds.

Step 4: $\mathbb{X}=\{M\mathcal{L}_\xi,\mathring{ð},\mathring{ð}',rV\}$. Commuting the original wave equation (5.31) with rV and using the commutator relation (2.41) for Y and V, one finds that $rV\varphi$ satisfies

$$\begin{aligned}&\left(2(r^2+a^2)YV+b_{V,rV}V+(b_{\phi,rV}+c_\phi)\mathcal{L}_\eta+(b_{0,rV}+c_0)\right)(rV\varphi)\\&\quad+\left(-2\,\mathring{ð}\,\mathring{ð}'-f_1(\theta)\mathcal{L}_\xi\mathcal{L}_\xi-f_2(\theta)\mathcal{L}_\eta\mathcal{L}_\xi-f_3(\theta)\mathcal{L}_\xi\right)(rV\varphi)-\vartheta_{rV}=0,\end{aligned}\tag{5.90}$$

where

$$b_{V,rV}=b_V+\frac{2(r^2+a^2)}{r},\quad b_{\phi,rV}=b_\phi-\frac{4ar}{r^2+a^2},\quad b_{0,rV}=b_0+1,\tag{5.91a}$$

$$\begin{aligned}\vartheta_{rV}&=rV\vartheta-\frac{(r^2-a^2)(\Delta-2Mr)}{r^2+a^2}YV\varphi-\frac{r\Delta}{2(r^2+a^2)}\partial_r(b_\phi+c_\phi)\mathcal{L}_\eta\varphi\\&\quad-\left(\frac{r\Delta}{2(r^2+a^2)}\partial_r(r^{-1}b_V)+\frac{4Mr}{r^2+a^2}-\frac{2r^2+a^2}{r^2}\right)(rV\varphi)-\frac{r\Delta}{2(r^2+a^2)}\partial_r(b_0+c_0)\varphi\end{aligned}\tag{5.91b}$$

and the c_ϕ, c_0, and f_i are again given in equation (5.32). The commuted wave equation (5.90) can be rewritten as

$$\widehat{\mathbb{S}}_s(rV\varphi)+b_{V,rV}V(rV\varphi)+b_{\phi,rV}\mathcal{L}_\eta(rV\varphi)+b_{0,rV}(rV\varphi)=\vartheta_{rV}.\tag{5.92}$$

The YV term in ϑ_{rV} can be expanded using the spin-weighted wave equation that φ is assumed to satisfy. Doing so, one finds that ϑ_{rV} is the sum of rV applied to ϑ and a sum of terms given by $O_\infty(1)$ coefficients multiplied by terms of the form either $\mathbb{S}^{\mathbf{a}}\varphi$ with $|\mathbf{a}|\leq 2$ or $rV\varphi$.

Again, this is in the form of equation (5.29) from the r^p lemma 5.5, and again, it is clear that the first two assumptions in lemma 5.5, on the asymptotics of b_V and b_ϕ, hold for the commuted equation (5.92) if they held for the original equation (5.29). The scalar $rV\varphi$ has the same spin as φ, and the condition on the leading-order coefficient in $b_{0,rV}$ is $0<b_{0,rV,0}+|s|+s$ $=b_{0,0}+|s|+s+1$, so that assumption (3) from lemma 5.3 holds for the commuted equation (5.92) if $0\leq b_{0,0}+|s|+s$ holds, which was assumption (3) for the original equation. In particular, if one starts with a spin-weighted wave equation of the form (5.29) that satisfies the three hypotheses

of the r^p lemma 5.5, then commuting with rV will give a new equation of the same form that also satisfies the three hypotheses.

Thus, for any multiindex $\mathbf{a}$, when considering $\{M\mathcal{L}_\xi, \mathring{\eth}, \mathring{\eth}', rV\}^{\mathbf{a}}$, there will be some number of operators from $\not{\mathbb{D}}$ and some number of compositions of rV. Since if φ satisfies the hypotheses of the r^p lemma 5.5, then so does $rV\varphi$, it follows by induction on the order of the composition of rV that each $\mathbb{X}^{\mathbf{a}}\varphi$ (where $\mathbb{X} = \{M\mathcal{L}_\xi, \mathring{\eth}, \mathring{\eth}', rV\}$) satisfies a spin-weighted wave equation satisfying the three hypotheses of the r^p lemma 5.5.

It remains to treat the corresponding ϑ terms. From applying rV, there is one term involving $rV\vartheta$ and additional terms of the form either $\not{\mathbb{D}}^{\mathbf{a}}\varphi$ with $|\mathbf{a}| \le 2$ or $rV\varphi$. Recall from step 2, the terms arising from commutation with $\mathring{\eth}'$ were either the $\mathring{\eth}'\vartheta$ or $\not{\mathbb{D}}^{\mathbf{a}}\varphi$ with $|\mathbf{a}| \le 2$, and similarly for $\mathring{\eth}$. Thus, from commuting $\{M\mathcal{L}_\xi, \mathring{\eth}, \mathring{\eth}', rV\}^{\mathbf{a}}$ through the spin-weighted wave equation (5.29), the terms that arise are of either of the form $\{M\mathcal{L}_\xi, \mathring{\eth}, \mathring{\eth}', rV\}^{\mathbf{a}}\vartheta$ or of the form $\{M\mathcal{L}_\xi, \mathring{\eth}, \mathring{\eth}', rV\}^{\mathbf{b}}\varphi$. All such $\{M\mathcal{L}_\xi, \mathring{\eth}, \mathring{\eth}', rV\}^{\mathbf{b}}\varphi$ arise from the additional terms in equation (5.91b) from commuting with rV, from the additional terms in equation (5.88) from commuting with $\mathring{\eth}$, or from the additional terms in equation (5.87) from commuting with $\mathring{\eth}'$. In commuting $\{M\mathcal{L}_\xi, \mathring{\eth}, \mathring{\eth}', rV\}^{\mathbf{a}}$ through the spin-weighted wave equations, the operators can at most once be applied so that they generate terms arising in one of the three equations (5.87), (5.88), or (5.91b), with all other factors either being applied to φ or to one of the coefficients. If the ϑ_{rV} equation (5.91b) is applied, then either the number of rV terms is reduced or the total order is reduced. If the $\vartheta_{\mathring{\eth}'}$ or $\vartheta_{\mathring{\eth}}$ equation (5.87) or (5.88) is applied, then the number of rV terms is unchanged, and either the number of angular derivatives is reduced or the total order is reduced. Thus, by applying a triple induction on total order, within that order of $\mathbb{S}$ derivatives, and within that order of $M\mathcal{L}_\xi$ derivatives, one obtains that the the r^p estimate (5.85) holds with $\mathbb{X} = \{M\mathcal{L}_\xi, \mathring{\eth}, \mathring{\eth}', rV\}$.

Step 5: $\mathbb{X} = \mathbb{D}$**.** In the domain of consideration $r \ge R_0 - M$, the operator MY can be expanded in terms of $M\mathcal{L}_\xi$, $\mathring{\eth}$, $\mathring{\eth}'$, rV and, conversely, the operator $M\mathcal{L}_\xi$ can be expanded in terms of MY, $\mathring{\eth}$, $\mathring{\eth}'$, rV. The coefficients appearing in these expansions are all at most $O_\infty(1)$, which implies the equivalence of the norms generated by these two sets of operators. To complete the proof, note that, on $\mathscr{I}^+$, rV vanishes on conformally regular functions and that $MY = 2M\mathcal{L}_\xi$. □

5.5. **Spin-weighted wave equations in the early region.** The following lemma allows norms on the hyperboloid Σ_t to be estimated in terms of norms on the hypersurface Σ_{init}, which extends to spacelike infinity. It is convenient to introduce first the following definition.

Lemma 5.7 (Controlling the early region)**.** *Under the same assumptions of Lemma 5.6, for any $k \in \mathbb{N}$, there are constants $\bar{R}_0 = \bar{R}_0(b_0, b_\phi, b_V)$ and $C = C(b_0, b_\phi, b_V)$ such that, if φ and ϑ are spin-weighted scalars satisfying (5.29), then, for all $R_0 \ge \bar{R}_0$, $\alpha \in [\delta, 2-\delta]$, and $t \le t_0$,*

$$
\begin{aligned}
&\|rV\varphi\|^2_{W^k_{\alpha-2}(\Sigma_t^{R_0})} + \|\varphi\|^2_{W^{k+1}_{-2}(\Sigma_t^{R_0})} \\
&+ \|\varphi\|^2_{W^{k+1}_{\alpha-3}(\Omega^{\text{early},R_0}_{\text{init},t})} + \|MY\varphi\|^2_{W^k_{-1-\delta}(\Omega^{\text{early},R_0}_{\text{init},t})} \\
&+ \|\varphi\|^2_{F^k(\mathscr{I}^+_{-\infty,t})} \\
&\le C\left(\|\varphi\|^2_{H^{k+1}_{\alpha-1}(\Sigma_{\text{init}})} + \|\varphi\|^2_{W^{k+1}_0(\Omega^{\text{early},R_0-M,R_0}_{\text{init},t})} + \|\varphi\|^2_{W^{k+1}_\alpha(\Sigma_t^{R_0-M,R_0})} + \|\vartheta\|^2_{W^k_{\alpha-3}(\Omega^{\text{early},R_0-M}_{\text{init},t})} \right).
\end{aligned}
\tag{5.93}
$$

Proof. Throughout this proof, $\lesssim$ is used to mean $\lesssim_{b_0,b_\phi,b_V}$, and we use mass normalization as in definition 4.4. The method for increasing the regularity that appeared in the proof of the higher regularity r^p lemma 5.6 applies in exactly the same way. Thus, it is sufficient to modify the proof of the original r^p lemma 5.5. The only change that must be made is in step 5, where the energy on the $\Sigma^{R_0-M}_{t_1}$ must be replaced by an energy on Σ_{init}. The energy densities $e_{i,X}$ can be estimated following the same ideas appearing in the step 5 of the proof of the r^p lemma 5.5. The major change is that on the Cauchy slice $\nu_a V^a \sim 1$ instead of $M^2 r^{-2}$. It remains the case that

$\nu_a\xi^a \sim 1 \sim \mathrm{d}\nu_a Y^a$. Thus, one finds

$$
\begin{aligned}
&\left| \int_{\Sigma_{\mathrm{init}}} \sum_{i=1}^{9} \sum_{X\in\{V,Y\}} \nu_a P^a_{i,X} \mathrm{d}^3\mu_\nu \right| \\
&\lesssim \int_{\Sigma_{\mathrm{init}}} \left(M^{-\alpha} r^\alpha |V\varphi|^2 + |Y\varphi|^2 + M^{-\alpha+2} r^{\alpha-2} |\mathring{\eth}' \varphi|^2 + M^{-\alpha+2} r^{\alpha-2}|\varphi|^2 \right) \mathrm{d}^3\mu \\
&\lesssim \int_{\Sigma_{\mathrm{init}}} \sum_{|\mathbf{a}|\leq 1} M^{-\alpha+2} r^{\alpha-2+2|\mathbf{a}|} |\mathbb{B}^{\mathbf{a}}\varphi|^2 \mathrm{d}^3\mu.
\end{aligned} \tag{5.94}
$$

The stated result now follows from the fact that, for any k,

$$
\int_{\Sigma_{\mathrm{init}}} \sum_{|\mathbf{a}|\leq 1} \sum_{|\mathbf{b}|\leq k} M^{-\alpha+2} r^{\alpha-2+2|\mathbf{a}|} |\mathbb{B}^{\mathbf{a}}\mathbb{D}^{\mathbf{b}}\varphi|^2 \mathrm{d}^3\mu \lesssim \int_{\Sigma_{\mathrm{init}}} \sum_{|\mathbf{a}|\leq k+1} M^{-\alpha+2} r^{\alpha-2+2|\mathbf{a}|} |\mathbb{B}^{\mathbf{a}}\varphi|^2 \mathrm{d}^3\mu, \tag{5.95}
$$

which completes the proof. □

6. The spin-weight −2 Teukolsky equation

In this section, we consider the field $\hat{\psi}_{-2}$ of spin-weight -2 that solves the Teukolsky equation (3.25a).

6.1. Extended system.

This section introduces a collection $\{\hat{\psi}_{-2}\}_{i=0}^4$ of conformally regular derivatives of $\hat{\psi}_{-2}$, a collection of rescalings $\{\hat{\varphi}^{(i)}_{-2}\}_{i=0}^4$ that are (depending on the index) divergent or vanishing at the horizon, shows that the $\hat{\varphi}^{(i)}_{-2}$ satisfy a system of wave equations, and finally shows that the $W^k_\alpha(\Sigma^{R_0}_{t_2})$ norms of the $\hat{\psi}^{(i)}_{-2}$ and $\hat{\varphi}^{(i)}_{-2}$ are equivalent for sufficiently large R_0.

Definition 6.1. Let $\hat{\psi}_{-2}$ be a scalar of spin-weight -2. Define

$$
\hat{\psi}^{(i)}_{-2} = \left(\frac{a^2+r^2}{M} V \right)^i \hat{\psi}_{-2}, \qquad 0 \leq i \leq 4. \tag{6.1}
$$

Definition 6.2. Let $\hat{\psi}_{-2}$ be a scalar of spin-weight -2 and $\{\hat{\psi}^{(i)}_{-2}\}_{i=0}^4$ be as in definition 6.1. Define

$$
\hat{\varphi}^{(0)}_{-2} = \left(\frac{\Delta}{r^2+a^2} \right)^2 \hat{\psi}^{(0)}_{-2}, \tag{6.2a}
$$

$$
\hat{\varphi}^{(i)}_{-2} = \frac{2}{M} \frac{(r^2+a^2)^2}{\Delta} V \hat{\varphi}^{(i-1)}_{-2} \qquad 1 \leq i \leq 4. \tag{6.2b}
$$

Remark 6.3. Compared to the quantity introduced by Ma in [33, Appendix A], which we denote here by $\hat{\phi}^{0,\mathrm{Ma}}_{-2}$, we have $\hat{\varphi}^{(0)}_{-2} = \sqrt{r^2+a^2}\bar{\kappa}_{1'}{}^2 \kappa_1{}^{-2} \hat{\phi}^{0,\mathrm{Ma}}_{-2}$ where the first factor $\sqrt{r^2+a^2}$ is to make the quantity nondegenerate at future null infinity and the other factor $\bar{\kappa}_{1'}{}^2\kappa_1{}^{-2}$ corresponds to a spin rotation of the frame.

Lemma 6.4. *If $\hat{\psi}_{-2}$ is a solution to (3.25a), then the variables $\hat{\varphi}^{(0)}_{-2}, \dots, \hat{\varphi}^{(4)}_{-2}$ satisfy the system*

$$
\widehat{\boxed{S}}_{-2} \begin{pmatrix} \hat{\varphi}^{(0)}_{-2} \\ \hat{\varphi}^{(1)}_{-2} \\ \hat{\varphi}^{(2)}_{-2} \\ \hat{\varphi}^{(3)}_{-2} \\ \hat{\varphi}^{(4)}_{-2} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \hat{\varphi}^{(0)}_{-2} \\ \hat{\varphi}^{(1)}_{-2} \\ \hat{\varphi}^{(2)}_{-2} \\ \hat{\varphi}^{(3)}_{-2} \\ \hat{\varphi}^{(4)}_{-2} \end{pmatrix} + \mathbf{B}\mathcal{L}_\eta \begin{pmatrix} \hat{\varphi}^{(0)}_{-2} \\ \hat{\varphi}^{(1)}_{-2} \\ \hat{\varphi}^{(2)}_{-2} \\ \hat{\varphi}^{(3)}_{-2} \\ \hat{\varphi}^{(4)}_{-2} \end{pmatrix} + \mathbf{C}V \begin{pmatrix} \hat{\varphi}^{(0)}_{-2} \\ \hat{\varphi}^{(1)}_{-2} \\ \hat{\varphi}^{(2)}_{-2} \\ \hat{\varphi}^{(3)}_{-2} \\ \hat{\varphi}^{(4)}_{-2} \end{pmatrix}, \tag{6.3}
$$

with

$$\mathbf{A} = \begin{pmatrix} -\frac{4r(M+r)}{a^2+r^2} & \frac{4M(Ma^2+a^2r-3Mr^2+r^3)}{(a^2+r^2)^2} & 0 & 0 & 0 \\ -\frac{6r(a^4+3Ma^2r+a^2r^2-Mr^3)}{M(a^2+r^2)^2} & \frac{2(a^4-12Ma^2r-2a^2r^2+4Mr^3-3r^4)}{(a^2+r^2)^2} & \frac{2M(Ma^2+a^2r-3Mr^2+r^3)}{(a^2+r^2)^2} & 0 & 0 \\ -\frac{6a^2(a^4+6Ma^2r-10Mr^3-r^4)}{M^2(a^2+r^2)^2} & -\frac{20a^2(Ma^2+a^2r-3Mr^2+r^3)}{M(a^2+r^2)^2} & \frac{2(a^4-12Ma^2r-2a^2r^2+4Mr^3-3r^4)}{(a^2+r^2)^2} & 0 & 0 \\ -\frac{12a^2(3Ma^4-2a^4r-24Ma^2r^2-2a^2r^3+5Mr^4)}{M^3(a^2+r^2)^2} & \frac{2a^2(-13a^4+82Ma^2r-30Mr^3+13r^4)}{M^2(a^2+r^2)^2} & -\frac{2(20Ma^4+17a^4r-69Ma^2r^2+17a^2r^3+3Mr^4)}{M(a^2+r^2)^2} & -\frac{4r(M+r)}{a^2+r^2} & 0 \\ \frac{24a^4(a^4+30Ma^2r-34Mr^3-r^4)}{M^4(a^2+r^2)^2} & \frac{128a^4(Ma^2+a^2r-3Mr^2+r^3)}{M^3(a^2+r^2)^2} & \frac{60a^2(-a^4+10Ma^2r-6Mr^3+r^4)}{M^2(a^2+r^2)^2} & -\frac{40a^2(Ma^2+a^2r-3Mr^2+r^3)}{M(a^2+r^2)^2} & -\frac{4(a^4-9Ma^2r+a^2r^2+7Mr^3)}{(a^2+r^2)^2} \end{pmatrix}, \tag{6.4a}$$

$$\mathbf{B} = -\frac{2a}{M^3(a^2+r^2)} \begin{pmatrix} 4M^3r & 0 & 0 & 0 & 0 \\ 3M^2(a^2-r^2) & 2M^3r & 0 & 0 & 0 \\ -12Ma^2r & 4M^2(a^2-r^2) & 0 & 0 & 0 \\ -12a^2(a^2-r^2) & -28Ma^2r & 3M^2(a^2-r^2) & -2M^3r & 0 \\ \frac{48a^4r}{M} & -40a^2(a^2-r^2) & -40Ma^2r & 0 & -4M^3r \end{pmatrix}, \tag{6.4b}$$

$$\mathbf{C} = -\frac{4(Ma^2+a^2r-3Mr^2+r^3)}{\Delta} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \tag{6.4c}$$

Proof. The rescaling in the variable $\hat{\varphi}_{-2}^{(0)}$ eliminates the Y terms of (3.25a) to yield the first row of the system. Repeated application of the commutator

$$\begin{aligned} \widehat{\boxed{\mathrm{S}}}_s\Big(\frac{(a^2+r^2)^2}{\Delta}V\varphi\Big) = {} & \frac{(a^2+r^2)^2}{\Delta}V\widehat{\boxed{\mathrm{S}}}_s(\varphi) + \frac{4ar}{a^2+r^2}\mathcal{L}_\eta\big(\frac{(a^2+r^2)^2}{\Delta}V\varphi\big) \\ & - \frac{4(Ma^2+a^2r-3Mr^2+r^3)}{\Delta}V\big(\frac{(a^2+r^2)^2}{\Delta}V\varphi\big) \\ & + \frac{a(a-r)(a+r)}{a^2+r^2}\mathcal{L}_\eta\varphi - \frac{2(a^4-10Ma^2r+6Mr^3-r^4)}{\Delta}V\varphi \\ & + \frac{(2Ma^4+a^4r-9Ma^2r^2+a^2r^3+Mr^4)\varphi}{(a^2+r^2)^2} \end{aligned} \tag{6.5}$$

gives the remaining rows. □

Lemma 6.5. *Let $\hat{\psi}_{-2}$ be a scalar of spin-weight -2 and $\{\hat{\psi}_{-2}^{(i)}\}_{i=0}^4$ be as in definition 6.1. Let $\{\hat{\varphi}_{-2}^{(i)}\}_{i=0}^4$ be as in definition 6.2. Let $k \in \mathbb{N}$, $\beta \in \mathbb{R}$, $R_0 \geq 10M$ and $0 \leq i \leq 4$. We have*

$$\sum_{i'=0}^{i} \|\hat{\psi}_{-2}^{(i')}\|_{W_\beta^k(\Sigma_t^{R_0})} \sim \sum_{i'=0}^{i} \|\hat{\varphi}_{-2}^{(i')}\|_{W_\beta^k(\Sigma_t^{R_0})}. \tag{6.6}$$

Furthermore, for $\alpha \in [0,2]$,

$$\begin{aligned} & \sum_{i'=0}^{i} \Big(\|rV\hat{\psi}_{-2}^{(i')}\|_{W_{\alpha-2}^{k-1}(\Sigma_t^{R_0})} + \|\hat{\psi}_{-2}^{(i')}\|_{W_{-2}^{k}(\Sigma_t^{R_0})}\Big) \\ & \quad \sim \sum_{i'=0}^{i} \Big(\|rV\hat{\varphi}_{-2}^{(i')}\|_{W_{\alpha-2}^{k-1}(\Sigma_t^{R_0})} + \|\hat{\varphi}_{-2}^{(i')}\|_{W_{-2}^{k}(\Sigma_t^{R_0})}\Big). \end{aligned} \tag{6.7}$$

Proof. The first step is to prove that, given $R_0 \geq 10M$ sufficiently large, each $\hat{\varphi}_{-2}^{(i)}$ is a linear combination of the $\hat{\psi}_{-2}^{(i')}$ with $0 \leq i' \leq i$ and coefficients that are analytic in R and vice versa. Let $\hat{V} = M^{-1}(r^2+a^2)V$ and extend this analytically through $R = 0$. First, observe that $\Delta^2(r^2+a^2)^{-1}$ and its inverse are analytic in R on intervals corresponding to $r \geq R_0$ and R not excessively negative. Second, observe that $\hat{\varphi}_{-2}^{(0)} = \Delta^2(r^2+a^2)^{-2}\hat{\psi}_{-2}$. Third, observe that $\hat{\psi}_{-2}^{(i)} = \hat{V}\hat{\psi}_{-2}^{(i-1)}$ and $\hat{\varphi}_{-2}^{(i)} = 2(\Delta(r^2+a^2)^{-1})^{-1}\hat{V}\hat{\varphi}_{-2}^{(i-1)}$ for $1 \leq i \leq 4$. Fourth, observe that if the operator $\hat{V}$ is applied to any function that is analytic in R on an interval extending through $R = 0$, then the result is also analytic in R on the same interval. The claim holds for $i = 0$ from the first two observations. From the third and fourth observations and induction, the claim follows for $1 \leq i \leq 4$.

Since the $\hat{\psi}_{-2}^{(i)}$ and $\hat{\varphi}_{-2}^{(i)}$ are linear combinations of each other with bounded coefficients (for $r \geq R_0$, which prevents the divergence or vanishing of powers of $\Delta(r^2+a^2)^{-1}$), it follows that, for any α,

$$\sum_{i'=0}^{i} \|\hat{\psi}_{-2}^{(i')}\|_{W_\alpha^0(\Sigma_t^{R_0})} \sim \sum_{i'=0}^{i} \|\hat{\varphi}_{-2}^{(i')}\|_{W_\alpha^0(\Sigma_t^{R_0})}. \tag{6.8}$$

Since, for any $\alpha \in \mathbb{R}$, the operators rV, Y, $\mathring{\eth}$, and $\mathring{\eth}'$ take $r^\alpha O_\infty(1)$ functions to $r^\alpha O_\infty(1)$ functions, the same estimate remains true when increasing the level of regularity from 0 to k. This proves estimate (6.6).

Now consider estimate (6.7). From estimate (6.6) with $\beta = -2$, there is the equivalence of $\sum_{i'=0}^{i}\|\hat{\psi}_{-2}^{(i')}\|^2_{W_{-2}^k(\Sigma_t^{R_0})}$ and $\sum_{i'=0}^{i}\|\hat{\varphi}_{-2}^{(i')}\|^2_{W_{-2}^k(\Sigma_t^{R_0})}$. Observe that if a prefactor f is analytic in $R = r^{-1}$, then its V derivative is $O_\infty(r^{-2})$ and rVf is $O_\infty(r^{-1})$. Thus, when considering $rV\sum_{i'=0}^{i}\hat{\psi}_{-2}^{(i')}$ and $rV\sum_{i'=0}^{i}\hat{\varphi}_{-2}^{(i')}$ the difference is bounded by a linear combination of the $\hat{\psi}_{-2}^{(i')}$ or of the $\hat{\varphi}_{-2}^{(i')}$ each with coefficients decaying like r^{-1}. Since $(\alpha-2)-2 \leq -2$, the lower-order terms arising from comparing $\sum_{i'=0}^{i}\|rV\hat{\psi}_{-2}^{(i')}\|_{W_{\alpha-2}^0(\Sigma_t^{R_0})}$ and $\sum_{i'=0}^{i}\|rV\hat{\varphi}_{-2}^{(i')}\|_{W_{\alpha-2}^0(\Sigma_t^{R_0})}$ are dominated by $\sum_{i'=0}^{i}\|\hat{\varphi}_{-2}^{(i')}\|_{W_{-2}^0(\Sigma_t^{R_0})} \sim \sum_{i'=0}^{i}\|\hat{\psi}_{-2}^{(i')}\|_{W_{-2}^0(\Sigma_t^{R_0})}$. The same holds after commuting with rV, Y, $\mathring{\eth}$, and $\mathring{\eth}'$, which completes the proof. □

6.2. **Basic energy and Morawetz (BEAM) condition.**

Definition 6.6. Let Σ be a smooth, achronal hypersurface. Let ν be a local map from Σ to $T\mathcal{M}$ such that ν is always normal to Σ. Let φ be a spin-weighted scalar field. Define

$$E_\Sigma^1(\varphi) = M\int_\Sigma \Big((\nu_a Y^a)|V\varphi|^2 + (\nu_a V^a)|Y\varphi|^2 + (\nu_a(V^a+Y^a))r^{-2}(|\mathring{\eth}\varphi|^2 + |\mathring{\eth}'\varphi|^2)\Big)\,\mathrm{d}^3\mu_\nu, \tag{6.9}$$

where $\mathrm{d}^3\mu_\nu$ denotes a Leray form as in definition 4.2. Further, for $k \in \mathbb{Z}^+$, let

$$E_\Sigma^k(\varphi) = \sum_{|\mathbf{a}|\leq k-1} M^{2|\mathbf{a}|}E_\Sigma^1(\mathbb{B}^{\mathbf{a}}\varphi). \tag{6.10}$$

Definition 6.7. Let $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$. Let φ be a spin-weighted scalar. Define

$$B_{t_1,t_2}^1(\varphi) = \int_{\Omega_{t_1,t_2}^{10M}} M^3 r^{-3}\sum_{|\mathbf{a}|=1}|\mathbb{B}^{\mathbf{a}}\varphi|^2\mathrm{d}^4\mu + \int_{\Omega_{t_1,t_2}} Mr^{-3}|\varphi|^2\mathrm{d}^4\mu. \tag{6.11}$$

Further, for $k \in \mathbb{Z}^+$, let

$$B_{t_1,t_2}^k(\varphi) = \sum_{|\mathbf{a}|\leq k-1} M^{2|\mathbf{a}|}B_{t_1,t_2}^1(\mathbb{B}^{\mathbf{a}}\varphi). \tag{6.12}$$

Definition 6.8 (BEAM condition for $\hat{\psi}_{-2}$)**.** Let $\hat{\psi}_{-2}$ be a scalar of spin-weight -2 and $\{\hat{\psi}_{-2}^{(i)}\}_{i=0}^2$ be as in definition 6.1. Assume $\hat{\psi}_{-2}$ satisfies the Teukolsky equation (3.25a). We shall say that the BEAM condition holds if for all sufficiently large $k \in \mathbb{N}$ and all $t_2 \geq t_1 \geq t_0$,

$$\sum_{i=0}^{2}\Big(E_{\Sigma_{t_2}}^k(\hat{\psi}_{-2}^{(i)}) + B_{t_1,t_2}^k(\hat{\psi}_{-2}^{(i)})\Big) \lesssim \sum_{i=0}^{2} E_{\Sigma_{t_1}}^k(\hat{\psi}_{-2}^{(i)}). \tag{6.13}$$

Definition 6.9 (Spin-weight -2 data norm on Σ_{t_0})**.** Let $\delta > 0$ be sufficiently small. Let $\hat{\psi}_{-2}$ be a scalar of spin-weight -2 and $\{\hat{\psi}_{-2}^{(i)}\}_{i=0}^{4}$ be as in definition 6.1. For $k \in \mathbb{Z}^+$, the initial data norm for $\hat{\psi}_{-2}$ on Σ_{init}, with regularity k is

$$\mathbb{I}_{-2}^{k} = \sum_{i=0}^{4}\left(\|\hat{\psi}_{-2}^{(i)}\|^2_{W_{-2}^{k}(\Sigma_{t_0})} + \|rV\hat{\psi}_{-2}^{(i)}\|^2_{W_{-\delta}^{k-1}(\Sigma_{t_0})}\right). \tag{6.14}$$

6.3. **Decay estimates.** This section proves three results. The first is the boundedness of various weighted norms. These bounds are proved using the r^p lemma 5.6. The second is a series of pointwise-in-t decay-estimates for various energies. The third gives improved rates of decay when $\mathcal{L}_{\xi}^{j}$ is applied. Because of the form of the BEAM assumption, the components $\hat{\psi}_{-2}^{(i)}$ for $i \in \{0, 1, 2\}$ are treated together. Further estimates are proved when $i = 3$ and then $i = 4$ are also included.

Lemma 6.10 (r^p estimate for $\hat{\psi}_{-2}^{(j)}$)**.** *Let $\delta > 0$ be sufficiently small. Let $\hat{\psi}_{-2}$ be a scalar of spin-weight -2 and $\{\hat{\psi}_{-2}^{(i)}\}_{i=0}^{4}$ be as in definition 6.1. Assume $\hat{\psi}_{-2}$ satisfies the Teukolsky equation (3.25a). For $i' \in \{0, \ldots, 4\}$, define $\ell(i') = \max(0, i' - 2)$. Let $i \in \{2, 3, 4\}$ and $\alpha \in [\delta, 2 - \delta]$. Assume the BEAM condition from definition 6.8 holds. If $k \in \mathbb{N}$ is sufficiently large, then for $t_2 \geq t_1 \geq t_0$, there is the bound*

$$\begin{aligned}
&\sum_{i'=0}^{i}\left(\|\hat{\psi}_{-2}^{(i')}\|^2_{W_{-2}^{k-\ell(i')}(\Sigma_{t_2})} + \|rV\hat{\psi}_{-2}^{(i')}\|^2_{W_{\alpha-2}^{k-1-\ell(i')}(\Sigma_{t_2})} + \|\hat{\psi}_{-2}^{(i')}\|^2_{W_{\alpha-3}^{k-1-\ell(i')}(\Omega_{t_1,t_2})}\right)\\
&\lesssim \sum_{i'=0}^{i}\left(\|\hat{\psi}_{-2}^{(i')}\|^2_{W_{-2}^{k-\ell(i')}(\Sigma_{t_1})} + \|rV\hat{\psi}_{-2}^{(i')}\|^2_{W_{\alpha-2}^{k-1-\ell(i')}(\Sigma_{t_1})}\right).
\end{aligned} \tag{6.15}$$

Proof. Consider the $\{\hat{\varphi}_{-2}^{(i)}\}_{i=0}^{4}$ which are defined in definition 6.2 and satisfy the 5-component system (6.3). The central idea in this proof is to apply the (higher-regularity) r^p lemma 5.6 to each component of the 5-component system (6.3). To do so, it is necessary to relate the components of the matrices of coefficients $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ in (6.3) to the coefficients b_V, b_η, b_0 in the hypotheses of the r^p lemma 5.6. The diagonal components of $\mathbf{A}$ all converge to nonpositive limits, so (when the corresponding $\hat{\varphi}_{-2}^{(i)}$ terms are moved from the right of the equation to the left) the condition $b_{0,0} + |s| + s \geq 0$ is always satisfied. The diagonal components of $\mathbf{B}$ are all $O_\infty(r^{-1})$. The diagonal components of $\mathbf{C}/r$ all converge to nonpositive limits, so the condition $b_{V,-1} \geq 0$ always holds. The off-diagonal components of the $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ couple each $\hat{\varphi}_{-2}^{(i')}$ to the other $\hat{\varphi}_{-2}^{(i)}$, which can be treated as inhomogeneities ϑ. There are no off-diagonal terms in $\mathbf{C}$, so these do not need to be treated. All the subdiagonal terms in $\mathbf{A}$ and $\mathbf{B}$ are $O_\infty(1)$. The only superdiagonal terms are the $(0,1)$ and $(1,2)$ components of $\mathbf{A}$, and these are both $O_\infty(r^{-1})$. Treating these as inhomogeneities, ϑ, will contribute $\|Mr^{-1}\hat{\varphi}_{-2}^{(i)}\|^2_{W_{\alpha-3}^{k}(\Omega_{t_1,t_2})}$ terms on the right with $i \in \{1, 2\}$; it is convenient to also add an $i = 0$ term to the right. In the r^p lemma 5.6, the $\mathscr{I}^+$ flux, and the space-time integrals of $Y\hat{\varphi}_{-2}^{(i)}$ are not needed to achieve the statement of the lemma and can be simply dropped. From all of this, the r^p lemma 5.6 implies, for each $i \in \{0, 1, 2, 3, 4\}$ and k, there are constants $R_0 \geq 10M$ and C [10] such that, with $\alpha \in [\delta, 2 - \delta]$,

$$\begin{aligned}
&\|rV\hat{\varphi}_{-2}^{(i)}\|^2_{W_{\alpha-2}^{k}(\Sigma_{t_2}^{R_0})} + \|\hat{\varphi}_{-2}^{(i)}\|^2_{W_{-2}^{k+1}(\Sigma_{t_2}^{R_0})} + \|\hat{\varphi}_{-2}^{(i)}\|^2_{W_{\alpha-3}^{k+1}(\Omega_{t_1,t_2}^{R_0})}\\
&\leq C\Bigg(\|rV\hat{\varphi}_{-2}^{(i)}\|^2_{W_{\alpha-2}^{k}(\Sigma_{t_1}^{R_0})} + \|\hat{\varphi}_{-2}^{(i)}\|^2_{W_{-2}^{k+1}(\Sigma_{t_1}^{R_0})}\\
&\qquad + \|\hat{\varphi}_{-2}^{(i)}\|^2_{W_{0}^{k+1}(\Omega_{t_1,t_2}^{R_0-M,R_0})} + \sum_{t\in\{t_1,t_2\}}\|\hat{\varphi}_{-2}^{(i)}\|^2_{W_{\alpha}^{k+1}(\Sigma_{t}^{R_0-M,R_0})}\\
&\qquad + \sum_{i'=0}^{i-1}\|\hat{\varphi}_{-2}^{(i')}\|^2_{W_{\alpha-3}^{k}(\Omega_{t_1,t_2}^{R_0-M})} + \sum_{i'=0}^{i-1}\|\mathcal{L}_\eta\hat{\varphi}_{-2}^{(i')}\|^2_{W_{\alpha-3}^{k}(\Omega_{t_1,t_2}^{R_0-M})} + \sum_{i'=0}^{2}\|Mr^{-1}\hat{\varphi}_{-2}^{(i')}\|^2_{W_{\alpha-3}^{k}(\Omega_{t_1,t_2}^{R_0-M})}\Bigg).
\end{aligned} \tag{6.16}$$

[10] In the applications of the r^p lemma 5.6 to each subequation of the system (6.3), the $\bar{R}_0$ and C for each i is different, but we can take R_0 and C stated here to be the maximum value among the sets of different $\bar{R}_0$ and different C, respectively, such that the estimate (6.16) holds for all $i \in \{0, 1, 2, 3, 4\}$.

From lemma 6.5, the $\hat{\varphi}^{(i)}_{-2}$ may be replaced by $\hat{\psi}^{(i)}_{-2}$. Furthermore, given an estimate of the form (6.16) for some i up to n, then, for $i = n+1$, one can control the terms involving the sum $\sum_{i'=0}^{i-1}$ by the previous estimates, at the expense of a further implicit constant. Furthermore, when making such a sum, for $i \geq 2$ and $\delta \leq \alpha \leq 2-\delta$, the integral over $\Omega^{R_0-M}_{t_1,t_2}$ in the final term can be divided into $\Omega^{R_0-M,R_0}_{t_1,t_2}$ and $\Omega^{R_0}_{t_1,t_2}$, with the integral over $\Omega^{R_0-M,R_0}_{t_1,t_2}$ absorbed into the other integral over $\Omega^{R_0-M,R_0}_{t_1,t_2}$, so that the final integral over $\Omega^{R_0-M}_{t_1,t_2}$ can be treated as an integral merely over $\Omega^{R_0}_{t_1,t_2}$. Thus, using the trivial bound $\|\hat{\psi}^{(i')}_{-2}\|^2_{W^k_{\alpha-3}(\Omega^{R_0}_{t_1,t_2})} \leq \|\hat{\psi}^{(i')}_{-2}\|^2_{W^{k+1}_{\alpha-3}(\Omega^{R_0}_{t_1,t_2})}$, one finds, for $i \in \{2,3,4\}$, there is a constant C such that

$$\begin{aligned}
&\sum_{i'=0}^{i}\left(\|rV\hat{\psi}^{(i')}_{-2}\|^2_{W^k_{\alpha-2}(\Sigma^{R_0}_{t_2})} + \|\hat{\psi}^{(i')}_{-2}\|^2_{W^{k+1}_{-2}(\Sigma^{R_0}_{t_2})} + \|\hat{\psi}^{(i')}_{-2}\|^2_{W^k_{\alpha-3}(\Omega^{R_0}_{t_1,t_2})}\right)\\
&\leq C\Bigg(\sum_{i'=0}^{i}\Bigg(\|rV\hat{\psi}^{(i')}_{-2}\|^2_{W^k_{\alpha-2}(\Sigma^{R_0}_{t_1})} + \|\hat{\psi}^{(i')}_{-2}\|^2_{W^{k+1}_{-2}(\Sigma^{R_0}_{t_1})}\\
&\qquad + \|\hat{\psi}^{(i')}_{-2}\|^2_{W^{k+1}_{\alpha}(\Omega^{R_0-M,R_0}_{t_1,t_2})} + \sum_{t\in\{t_1,t_2\}}\|\hat{\psi}^{(i')}_{-2}\|^2_{W^{k+1}_{\alpha}(\Sigma^{R_0-M,R_0}_{t})}\Bigg)\\
&\qquad + \sum_{i'=0}^{2}\|Mr^{-1}\hat{\psi}^{(i')}_{-2}\|^2_{W^k_{\alpha-3}(\Omega^{R_0}_{t_1,t_2})}\Bigg).
\end{aligned}\tag{6.17}$$

Consider $i = 2$. Recall that the implicit constant in the bound (6.17) is independent of R_0. Thus, for $i = 2$, by taking R_0 sufficiently large, Mr^{-1} can be taken sufficiently small relative to the implicit constant, and the $\|Mr^{-1}\hat{\psi}^{(i)}_{-2}\|^2_{W^k_{\alpha-3}(\Omega^{R_0}_{t_1,t_2})}$ terms on the right can be absorbed into the $\|\hat{\psi}^{(i)}_{-2}\|^2_{W^k_{\alpha-3}(\Omega^{R_0}_{t_1,t_2})}$ terms on the left. Because the energy $\sum_{i'=0}^{2} E^k_{\Sigma_t}(\hat{\psi}^{(i')}_{-2})$ controls all derivatives, and because for $r \leq R_0$, there is a constant $C(R_0,p)$ such that $1 \leq r^p \leq C(R_0,p)$, one finds that, for any β, there is the bound $\|\hat{\psi}^{(i)}_{-2}\|^2_{W^k_\beta(\Sigma^{r_+,R_0}_t)} \lesssim C(R_0,\beta)E^k_{\Sigma^{r_+,R_0}_t}(\hat{\psi}^{(i)}_{-2})$. Similarly, for $i = 2$, the integrals over $\Omega^{r_+,R_0}_{t_1,t_2}$ in the bound (6.17) can be controlled by $\sum_{i=0}^{2} E^k_{\Sigma_{t_1}}(\hat{\psi}^{(i)}_{-2})$ if the BEAM condition from definition 6.8 holds. Thus, under these conditions, the claim of the lemma, inequality (6.15), holds for $i = 2$.

A similar argument holds for $i \in \{3,4\}$; however, it is no longer true that the energy appearing in the BEAM condition, $\sum_{i'=0}^{2} E^k_{\Sigma^{r_+,R_0}_t}(\hat{\psi}^{(i')}_{-2})$, controls $\|\hat{\psi}^{(i)}_{-2}\|^2_{W^k_\beta(\Sigma^{r_+,R_0}_t)}$. To overcome this, one can apply $rV \in \mathbb{D}$, so that, with $i \in \{3,4\}$, $\ell = i-2$, for any k,β,

$$\|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k-\ell}_\beta(\Sigma^{r_+,R_0}_t)} \lesssim_{R_0,\beta} \sum_{i'=0}^{2} E^k_{\Sigma^{r_+,R_0}_t}(\hat{\psi}^{(i')}_{-2}),\tag{6.18}$$

and similarly for the spacetime integral over $\Omega^{r_+,R_0}_{t_1,t_2}$ if the BEAM condition from definition 6.8 holds. Furthermore,

$$\sum_{i'=0}^{i}\|\hat{\psi}^{(i')}_{-2}\|^2_{W^{k-\ell(i')}_\beta(\Sigma^{r_+,R_0}_t)} \sim_{R_0,\beta} \sum_{i'=0}^{2} E^k_{\Sigma^{r_+,R_0}_t}(\hat{\psi}^{(i')}_{-2}),\tag{6.19}$$

which is needed at $t = t_1$. From these and the previous arguments, inequality (6.15) holds for $i = \{3,4\}$. ∎

Lemma 6.11 (Decay estimates for $\hat{\psi}^{(i)}_{-2}$ with $i \in \{2,3,4\}$)**.** *Let $\delta > 0$ be sufficiently small. Let $\hat{\psi}_{-2}$ be a scalar of spin-weight -2 and $\{\hat{\psi}^{(i)}_{-2}\}_{i=0}^4$ be as in definition 6.1. Assume $\hat{\psi}_{-2}$ satisfies the Teukolsky equation (3.25a). For $i' \in \{0,\dots,4\}$, define $\ell(i') = \max(0, i'-2)$. Let $i \in \{2,3,4\}$ and $\alpha \in [\delta, 2-\delta]$. Assume the BEAM condition from definition 6.8 holds. If $k \in \mathbb{N}$ is sufficiently* *large, then for $t \geq t_0$, there is the bound*

$$\sum_{i'=0}^{i}\Big(\|\hat{\psi}_{-2}^{(i')}\|^2_{W_{-2}^{k+2(i-5)-\ell(i')}(\Sigma_t)} + \|rV\hat{\psi}_{-2}^{(i')}\|^2_{W_{\alpha-2}^{k+2(i-5)-\ell(i')-1}(\Sigma_t)} + \|\hat{\psi}_{-2}^{(i')}\|^2_{W_{\alpha-3}^{k+2(i-5)-\ell(i')-1}(\Omega_{t,\infty})}\Big) \lesssim t^{\alpha-10+9\delta+(2-2\delta)i}\mathbb{I}^k_{-2}. \tag{6.20}$$

Proof. The strategy of the proof is to apply the pigeonhole lemma 5.2 to the r^p bound (6.15). Let

$$F^i(k,\alpha,t) = \sum_{i'=0}^{i}\Big(\|\hat{\psi}_{-2}^{(i')}\|^2_{W_{-2}^{k-\ell(i')}(\Sigma_t)} + \|rV\hat{\psi}_{-2}^{(i')}\|^2_{W_{\alpha-2}^{k-\ell(i')-1}(\Sigma_t)}\Big) \tag{6.21}$$

for $\alpha \geq \delta$ and $F(k,\alpha,t) = 0$ for $\alpha < \delta$. Here the i denotes how many of the $\hat{\psi}_{-2}^{(i')}$ are to be treated, k the level of regularity, and α the weight.

Observe that, since rV is in the set of operators used to define regularity $\mathbb{D}$, and since $(\alpha+1)-3 \geq -2$, one has that

$$\|\hat{\psi}_{-2}^{(i)}\|^2_{W_{(\alpha+1)-3}^{(k+1)-1}(\Omega_{t_1,t_2})} \gtrsim \int_{t_1}^{t_2}\Big(\|\hat{\psi}_{-2}^{(i)}\|^2_{W^k_{-2}(\Sigma_t)} + \|rV\hat{\psi}_{-2}^{(i)}\|^2_{W^{k-1}_{\alpha-2}(\Sigma_t)}\Big)\mathrm{d}t, \tag{6.22}$$

Thus, the r^p bound (6.15) can be written in the form

$$F^i(k,\alpha,t_2) + M^{-1}\int_{t_1}^{t_2}F^i(k-1,\alpha-1,t)\mathrm{d}t \lesssim F^i(k,\alpha,t_1) \tag{6.23}$$

for $\alpha \in [\delta, 2-\delta]$. This hierarchy of estimates is in the form treated by the pigeonhole lemma 5.2, and the assumptions (1) and (2) in the pigeonhole lemma 5.2 are easily seen to be satisfied from lemma 5.3.

Consider first the case $i = 4$. From applying the pigeonhole lemma 5.2 to the hierarchy (6.23), one finds $F^4(k-2,\alpha,t) \lesssim t^{\alpha-2+\delta}F^4(k,2-\delta,t_0)$. Applying this decay estimate and the r^p bound (6.15) a second time, one obtains the bound

$$\sum_{i'=0}^{4}\|\hat{\psi}_{-2}^{(i')}\|^2_{W_{\alpha-3}^{k-2-\ell(i')-1}(\Omega_{t,\infty})} \lesssim t^{\alpha-2+\delta}F^4(k,2-\delta,t_0). \tag{6.24}$$

A third application shows that $F^4(k,2-\delta,t_0)$ is bounded by $\mathbb{I}^k_{-2}$. This proves the desired inequality (6.20) in the case $i = 4$.

Consider now lower i. Observing that $\hat{\psi}_{-2}^{(i+1)} = M^{-1}(r^2+a^2)V\hat{\psi}_{-2}^{(i)}$, $r^2+a^2 = r^2O_\infty(1)$, one finds

$$\|rV\hat{\psi}_{-2}^{(i)}\|^2_{W_{-\delta}^{k-l(i)-1}(\Sigma_t)} \leq \|\hat{\psi}_{-2}^{(i+1)}\|^2_{W_{-2-\delta}^{k-l(i+1)}(\Sigma_t)}. \tag{6.25}$$

Additionally, one also has the trivial estimate

$$\sum_{i'=0}^{i}\|\hat{\psi}_{-2}^{(i')}\|^2_{W_{-2}^{k-l(i')}(\Sigma_t)} \leq \sum_{i'=0}^{i+1}\|\hat{\psi}_{-2}^{(i')}\|^2_{W_{-2}^{k-l(i')}(\Sigma_t)}. \tag{6.26}$$

Thus, one finds $F^i(k,2-\delta,t) \lesssim F^{i+1}(k,\delta,t)$. In particular, $F^3(k-2,2-\delta,t) \lesssim F^4(k-2,\delta,t) \lesssim t^{-2+2\delta}\mathbb{I}^k_{-2}$. Applying the pigeonhole lemma 5.2 that treats hierarchies where the top energy is known to decay a priori, one finds $F^3(k-4,\alpha,t) \leq t^{\alpha-(2-\delta)-(2-2\delta)}\mathbb{I}^k_{-2}$. The spacetime integral and estimate by the energy of the initial data are estimated in the same way as in the $i=4$ case, which proves inequality (6.20) in the case $i = 3$. Observing $F^2(k-4,2-\delta,t) \lesssim F^3(k-4,\delta,t) \lesssim t^{\alpha-4+3\delta}\mathbb{I}^k_{-2}$ and iterating the same argument once more proves inequality (6.20) in the case $i = 2$. ∎

Lemma 6.12 (Decay estimates for $\mathcal{L}_\xi^j\hat{\psi}_{-2}^{(i')}$, $i' \in \{2,3,4\}$)**.** *Let $\delta > 0$ be sufficiently small. Let $\hat{\psi}_{-2}$ be a scalar of spin-weight -2 and $\{\hat{\psi}_{-2}^{(i)}\}_{i=0}^4$ be as in definition 6.1. Assume $\hat{\psi}_{-2}$ satisfies the Teukolsky equation (3.25a). For $i, i' \in \{0,\dots,4\}$, define $\ell(i,i') = 2(i-5) - \max(0, i'-2)$. Let $i \in \{2,3,4\}$ and $\delta \leq \alpha \leq 2-\delta$. Assume the BEAM condition from definition 6.8 holds.*

(1) *If $k,j\in\mathbb{N}$ are such that $k-3j$ is sufficiently large, then there are the energy and Morawetz estimates for $t\geq t_0$,*

$$\sum_{i'=0}^{i}\Big(\|\mathcal{L}_\xi^j\hat\psi_{-2}^{(i')}\|^2_{W_{-2}^{k-3j-\ell(i,i')}(\Sigma_t)}+\|r\mathcal{L}_\xi^jV\hat\psi_{-2}^{(i')}\|^2_{W_{\alpha-2}^{k-3j-\ell(i,i')-1}(\Sigma_t)}+\|\mathcal{L}_\xi^j\hat\psi_{-2}^{(i')}\|^2_{W_{\alpha-3}^{k-3j-\ell(i,i')-1}(\Omega_{t,\infty})}\Big)$$
$$\lesssim t^{\alpha-10+9\delta+(2-2\delta)i-(2-2\delta)j}\mathbb{I}^k_{-2}. \tag{6.27}$$

(2) *If $k,j\in\mathbb{N}$ are such that $k-3j$ is sufficiently large, then there are the pointwise decay estimates for $t\geq t_0$*

$$\sum_{i'=0}^{i}|\mathcal{L}_\xi^j\hat\psi_{-2}^{(i')}|_{k-3j-\ell(i,i')-7,\mathbb{D}}\lesssim rv^{-1}t^{-(1-\delta)(\frac{9}{2}+j-i)+\delta}(\mathbb{I}^k_{-2})^{1/2}. \tag{6.28}$$

Proof. Observe that $\mathcal{L}_\xi$ is a symmetry of the Teukolsky equation (3.25a). Furthermore, if $\hat\psi_{-2}$ is replaced by $\mathcal{L}_\xi^j\hat\psi_{-2}$, then the $\{\hat\psi_{-2}^{(i)}\}$ in definition 6.1 are replaced by $\mathcal{L}_\xi^j\hat\psi_{-2}^{(i)}$. From the r^p estimate (6.15) for $\mathcal{L}_\xi^j\hat\psi_{-2}$, one has for $\delta\leq\alpha\leq2-\delta$, $j,k\in\mathbb{N}$, and $i\in\{2,3,4\}$,

$$\sum_{i'=0}^{i}\Big(\|\mathcal{L}_\xi^j\hat\psi_{-2}^{(i')}\|^2_{W_{-2}^{k-\ell(i')}(\Sigma_{t_2})}+\|rV\mathcal{L}_\xi^j\hat\psi_{-2}^{(i')}\|^2_{W_{\alpha-2}^{k-1-\ell(i')}(\Sigma_{t_2})}+\|\mathcal{L}_\xi^j\hat\psi_{-2}^{(i')}\|^2_{W_{\alpha-3}^{k-1-\ell(i')}(\Omega_{t_1,t_2})}\Big)$$
$$\lesssim\sum_{i'=0}^{i}\left(\|\mathcal{L}_\xi^j\hat\psi_{-2}^{(i')}\|^2_{W_{-2}^{k-\ell(i')}(\Sigma_{t_1})}+\|rV\mathcal{L}_\xi^j\hat\psi_{-2}^{(i')}\|^2_{W_{\alpha-2}^{k-1-\ell(i')}(\Sigma_{t_1})}\right). \tag{6.29}$$

Similarly, the basic decay lemma 6.11 gives

$$\sum_{i'=0}^{i}\Big(\|\mathcal{L}_\xi^j\hat\psi_{-2}^{(i')}\|^2_{W_{-2}^{k+2(i-5)-\ell(i')}(\Sigma_t)}+\|rV\mathcal{L}_\xi^j\hat\psi_{-2}^{(i')}\|^2_{W_{\alpha-2}^{k+2(i-5)-\ell(i')-1}(\Sigma_t)}+\|\mathcal{L}_\xi^j\hat\psi_{-2}^{(i')}\|^2_{W_{\alpha-3}^{k+2(i-5)-\ell(i')-1}(\Omega_{t,\infty})}\Big)$$
$$\lesssim t^{\alpha-10+9\delta+(2-2\delta)i}\mathbb{I}^{k+j}_{-2}. \tag{6.30}$$

Rearranging the expansions (2.40a)-(2.40b) for V and Y, for r sufficiently large, one can write Y as a weighted sum of $\mathcal{L}_\xi$, V, and $r^{-2}\mathcal{L}_\eta$ all with $O_\infty(1)$ coefficients. Using this to eliminate Y from the Teukolsky equation (3.25a), recaling equation (2.36e) for $\widehat{\square}$, and isolating the term $r^2V\mathcal{L}_\xi\hat\psi_{-2}$, one can write $r^2V\mathcal{L}_\xi\hat\psi_{-2}$ as a weighted sum of $(rV)^2\hat\psi_{-2}$, $rV\hat\psi_{-2}$, $r^{-1}\mathcal{L}_\eta(rV)\hat\psi_{-2}$, $r^{-1}\mathcal{L}_\eta\hat\psi_{-2}$, $S_s\hat\psi_{-2}$, $\mathcal{L}_\xi\hat\psi_{-2}$, and $\hat\psi_{-2}$ all with $O_\infty(1)$ coefficients. Rewriting $\mathcal{L}_\xi$ again as a weighted sum of Y, V, and $r^{-2}\mathcal{L}_\eta$ all with $O_\infty(1)$ coefficients, one finds that $r^2V\mathcal{L}_\xi\hat\psi_{-2}$ can be written as a linear combination with $O_\infty(1)$ coefficients of $r^{-1}\mathcal{L}_\eta(rV\hat\psi_{-2})$ and terms of the form $X_2X_1\hat\psi_{-2}$ with $X_1,X_2\in\mathbb{D}\cup\{1\}$. The commutator of the operator $M^{-1}(r^2+a^2)V$ used to construct the $\hat\psi_{-2}^{(i)}$ with any of the operators rV, $\mathcal{L}_\xi$, $\mathcal{L}_\eta$, S_s, 1 appearing in the expansion of the $r^2V\mathcal{L}_\xi\hat\psi_{-2}$ is in the span of $M^{-1}(r^2+a^2)V$, rV, and 1. Thus, induction implies that a similar expansion exists for each of the $r^2V\hat\psi_{-2}^{(i)}$, but also involving the previous $\hat\psi_{-2}^{(i')}$ with $i'<i$. Thus,

$$\begin{aligned}\sum_{i'=0}^{i}\|rV\mathcal{L}_\xi^{j+1}\hat\psi_{-2}^{(i')}\|^2_{W_{-\delta}^{k-2-\ell(i')}(\Sigma_t)}&\lesssim\sum_{i'=0}^{i}\|r^2V\mathcal{L}_\xi^{j+1}\hat\psi_{-2}^{(i')}\|^2_{W_{-\delta-2}^{k-2-\ell(i')}(\Sigma_t)}\\&\lesssim\sum_{i'=0}^{i}\|\mathcal{L}_\xi^{j}\hat\psi_{-2}^{(i')}\|^2_{W_{-\delta-2}^{k-\ell(i')}(\Sigma_t)}\\&\lesssim\sum_{i'=0}^{i}\|\mathcal{L}_\xi^{j}\hat\psi_{-2}^{(i')}\|^2_{W_{-2}^{k-\ell(i')}(\Sigma_t)}.\end{aligned} \tag{6.31}$$

Since $\mathcal{L}_\xi$ is a linear combination of Y, V and $r^{-2}\mathcal{L}_\eta$ with $O_\infty(1)$ coefficients, one also finds

$$\sum_{i'=0}^{i}\|\mathcal{L}_\xi^{j+1}\hat\psi_{-2}^{(i')}\|^2_{W_{-2}^{k-1-\ell(i')}(\Sigma_t)}\lesssim\sum_{i'=0}^{i}\|\mathcal{L}_\xi^{j}\hat\psi_{-2}^{(i')}\|^2_{W_{-2}^{k-\ell(i')}(\Sigma_t)}. \tag{6.32}$$

Combining these results, one finds

$$\sum_{i'=0}^{i}\|\mathcal{L}_\xi^{j+1}\hat{\psi}_{-2}^{(i')}\|^2_{W^{k-1-\ell(i')}_{-2}(\Sigma_t)}+\sum_{i'=0}^{i}\|rV\mathcal{L}_\xi^{j+1}\hat{\psi}_{-2}^{(i')}\|^2_{W^{k-2-\ell(i')}_{-\delta}(\Sigma_t)}\lesssim\sum_{i'=0}^{i}\|\mathcal{L}_\xi^{j}\hat{\psi}_{-2}^{(i')}\|^2_{W^{k-\ell(i')}_{-2}(\Sigma_t)}. \tag{6.33}$$

With these preliminaries proved, one can now consider the proof of the energy and Morawetz estimate (6.27). The $j=0$ case is proved in lemma 6.11. If inequality (6.27) is known to hold for j, then inequality (6.33) implies for $i\in\{2,3,4\}$

$$\begin{aligned}\sum_{i'=0}^{i}\|\mathcal{L}_\xi^{j+1}\hat{\psi}_{-2}^{(i')}\|^2_{W^{k+2(i-5)-3j-1-\ell(i')}_{-2}(\Sigma_t)}+\sum_{i'=0}^{i}\|rV\mathcal{L}_\xi^{j+1}\hat{\psi}_{-2}^{(i')}\|^2_{W^{k+2(i-5)-3j-2-\ell(i')}_{-\delta}(\Sigma_t)}&\\ \lesssim t^{-10+10\delta+(2-2\delta)i-(2-2\delta)j}\mathbb{I}^k_{-2}.&\end{aligned} \tag{6.34}$$

The hierarchy (6.29) and the bound at the top of the hierarchy (6.34) provide the hypotheses necessary to apply the pigeonhole lemma 5.2, an application of which implies

$$\begin{aligned}\sum_{i'=0}^{i}\|\mathcal{L}_\xi^{j+1}\hat{\psi}_{-2}^{(i')}\|^2_{W^{k+2(i-5)-3j-3-\ell(i')}_{-2}(\Sigma_t)}+\sum_{i'=0}^{i}\|rV\mathcal{L}_\xi^{j+1}\hat{\psi}_{-2}^{(i')}\|^2_{W^{k+2(i-5)-3j-4-\ell(i')}_{\alpha-2}(\Sigma_t)}&\\ \lesssim t^{\alpha-12+11\delta+(2-2\delta)i-(2-2\delta)j}\mathbb{I}^k_{-2}.&\end{aligned} \tag{6.35}$$

Writing $-12+11\delta-(2-2\delta)j=-10+9\delta-(2-2\delta)(j+1)$, one obtains inequality (6.27) for $j+1$, so inequality (6.27) holds for all $j\in\mathbb{N}$ by induction.

From the Sobolev inequality (4.49) with $\gamma=\delta$ and the energy estimate (6.27) with $\alpha=1+\delta$ and $\alpha=1-\delta$, one finds

$$\sum_{i'=0}^{i}|\mathcal{L}_\xi^{j}\hat{\psi}_{-2}^{(i')}|^2_{k'-3,\mathbb{D}}\lesssim t^{-(1-\delta)(9+2j-2i)}\mathbb{I}^k_{-2}. \tag{6.36}$$

Alternatively, having already established the limit as $t\to\infty$ is zero, one can now apply the anisotropic spacetime Sobolev inequality (4.53). Applying this, the trivial bound $-3<-3+\delta$, and the Morawetz estimate (6.27) with $\alpha=\delta$, one finds

$$\begin{aligned}\sum_{i'=0}^{i}|\mathcal{L}_\xi^{j}r^{-1}\hat{\psi}_{-2}^{(i')}|^2_{k'-7,\mathbb{D}}&\lesssim\sum_{i'=0}^{i}\|\mathcal{L}_\xi^{j}r^{-1}\hat{\psi}_{-2}^{(i')}\|^{1/2}_{W^{k'-4}_{-1}(\Omega_{t,\infty})}\|\mathcal{L}_\xi^{j+1}r^{-1}\hat{\psi}_{-2}^{(i')}\|^{1/2}_{W^{k'-4}_{-1}(\Omega_{t,\infty})}\\&\lesssim\sum_{i'=0}^{i}\|\mathcal{L}_\xi^{j}\hat{\psi}_{-2}^{(i')}\|^{1/2}_{W^{k'-4}_{-3}(\Omega_{t,\infty})}\|\mathcal{L}_\xi^{j+1}\hat{\psi}_{-2}^{(i')}\|^{1/2}_{W^{k'-4}_{-3}(\Omega_{t,\infty})}\\&\lesssim\sum_{i'=0}^{i}\|\mathcal{L}_\xi^{j}\hat{\psi}_{-2}^{(i')}\|^{1/2}_{W^{k'-4}_{-3+\delta}(\Omega_{t,\infty})}\|\mathcal{L}_\xi^{j+1}\hat{\psi}_{-2}^{(i')}\|^{1/2}_{W^{k'-4}_{-3+\delta}(\Omega_{t,\infty})}\\&\lesssim t^{-(1-\delta)(11+2j-2i)}\mathbb{I}^k_{-2}.\end{aligned} \tag{6.37}$$

Combining the two pointwise estimates and observing $v^{-1}\lesssim\min(r^{-1},t^{-1})$ gives the desired estimate (6.28). ☐

6.4. **Improved decay estimates.** In this section, we build on the results in lemma 6.12 and improve the decay estimates for $\mathcal{L}_\xi^{j}\hat{\psi}_{-2}^{(i)}$ for $i\in\{0,1\}$ in exterior region (where $r\geq t$) and interior region (where $r<t$), respectively. This is done by rewriting the first two lines of (6.3) as an elliptic equation of $\hat{\varphi}_{-2}^{(i)}$ with source terms each of which either contains at least one $\mathcal{L}_\xi$ derivative (which has extra $t^{-1+\delta}$ decay from lemma 6.12) or have an extra r^{-1} prefactor. We exploit this extra $t^{-1+\delta}$ decay and r^{-1} prefactor in the source terms, and an elliptic estimate yields improved pointwise-in-t decay estimates for $\mathcal{L}_\xi^{j}\hat{\psi}_{-2}^{(i)}$ ($i=0,1$) and their spacetime norms in different regions.

The decay estimates for all $\mathcal{L}_\xi^{j}\hat{\psi}_{-2}^{(i)}$ ($i=0,\dots,4$) are as follows.

Theorem 6.13 (Decay estimates with improvements for $\hat{\psi}_{-2}^{(i)}$ for $i\in\{0,1\}$)**.** *Let $\delta>0$ be sufficiently small. Let $\hat{\psi}_{-2}$ be a scalar of spin-weight -2 and $\{\hat{\psi}_{-2}^{(i)}\}_{i=0}^{4}$ be as in definition 6.1. Assume*

$\hat{\psi}_{-2}$ satisfies the Teukolsky equation (3.25a). Assume the BEAM condition from definition 6.8 holds. There is a regularity constant K such that the following holds. If $k, j \in \mathbb{N}$ are such that $k-3j-K \geq 0$, then with $k'' = k-3j-K$,

(1) *In the exterior region where $r \geq t$, we have for $i \in \{0, \dots, 4\}$ and $\delta \leq \alpha \leq 2-\delta$ the energy and Morawetz estimates for $t \geq t_0$*

$$\sum_{i'=0}^{i} \Big(\|\mathcal{L}_\xi^j \hat{\psi}_{-2}^{(i')}\|^2_{W^{k''}_{-2}(\Sigma_t^{\text{ext}})} + \|r\mathcal{L}_\xi^j V \hat{\psi}_{-2}^{(i')}\|^2_{W^{k''-1}_{\alpha-2}(\Sigma_t^{\text{ext}})} + \|\mathcal{L}_\xi^j \hat{\psi}_{-2}^{(i')}\|^2_{W^{k''-1}_{\alpha-3}(\Omega_{t,\infty}^{\text{ext}})} \Big) \\ \lesssim t^{\alpha-10+9\delta+(2-2\delta)i-(2-2\delta)j} \mathbb{I}^k_{-2}, \tag{6.38}$$

and pointwise decay estimates for $t \geq t_0$

$$\sum_{i'=0}^{i} |\mathcal{L}_\xi^j \hat{\psi}_{-2}^{(i')}|_{k'',\mathbb{D}} \lesssim r v^{-1} t^{-(1-\delta)(\frac{9}{2}+j-i)+\delta} (\mathbb{I}^k_{-2})^{1/2}. \tag{6.39}$$

(2) *In the interior region where $r < t$, for $i \in \{2,3,4\}$, and $\delta \leq \alpha \leq 2-\delta$, there are the energy and Morawetz estimates for $t \geq t_0$*

$$\sum_{i'=0}^{i} \Big(\|\mathcal{L}_\xi^j \hat{\psi}_{-2}^{(i')}\|^2_{W^{k''}_{-2}(\Sigma_t^{\text{int}})} + \|r\mathcal{L}_\xi^j V \hat{\psi}_{-2}^{(i')}\|^2_{W^{k''-1}_{\alpha-2}(\Sigma_t^{\text{int}})} + \|\mathcal{L}_\xi^j \hat{\psi}_{-2}^{(i')}\|^2_{W^{k''-1}_{\alpha-3}(\Omega_{t,\infty}^{\text{int}})} \Big) \\ \lesssim t^{\alpha-10+9\delta+(2-2\delta)i-(2-2\delta)j} \mathbb{I}^k_{-2}, \tag{6.40}$$

and pointwise decay estimates for $t \geq t_0$

$$\sum_{i'=0}^{i} |\mathcal{L}_\xi^j \hat{\psi}_{-2}^{(i')}|_{k'',\mathbb{D}} \lesssim r v^{-1} t^{-(1-\delta)(\frac{9}{2}+j-i)+\delta} (\mathbb{I}^k_{-2})^{1/2}. \tag{6.41}$$

Moreover, we have for $t \geq t_0$ that

$$|\mathcal{L}_\xi^j \hat{\psi}_{-2}|_{k'',\mathbb{D}} \lesssim r^{-1+2\delta} v^{-1} t^{-(1-\delta)(\frac{5}{2}+j)+\delta} (\mathbb{I}^k_{-2})^{1/2}, \tag{6.42a}$$

$$\|\mathcal{L}_\xi^j \hat{\psi}_{-2}\|^2_{W^{k''}_{\alpha+1-3\delta}(\Omega_{t,\infty}^{\text{int}})} \lesssim t^{-(6+2j)(1-\delta)+\alpha} \mathbb{I}^k_{-2}, \tag{6.42b}$$

$$|\mathcal{L}_\xi^j \hat{\psi}_{-2}^{(1)}|_{k'',\mathbb{D}} \lesssim r^{\delta} v^{-1} t^{-(1-\delta)(\frac{5}{2}+j)+\delta} (\mathbb{I}^k_{-2})^{1/2}, \tag{6.42c}$$

$$\|\mathcal{L}_\xi^j \hat{\psi}_{-2}^{(1)}\|^2_{W^{k''}_{\alpha-1-\delta}(\Omega_{t,\infty}^{\text{int}})} \lesssim t^{-(6+2j)(1-\delta)+\alpha} \mathbb{I}^k_{-2}. \tag{6.42d}$$

Proof. We prove point (1) first. Note that the estimates (6.38) and (6.39) have been proven for $i = 2,3,4$ in lemma 6.12. In the $i = 0,1$ cases, these estimates improve the pointwise-in-t decay compared to the pointwise estimate (6.28) and Morawetz estimate (6.27), hence they hold trivially if $r \leq 10M$ since then $t \leq 10M$ is finite. Therefore, we shall only consider the exterior region intersected with $r \geq 10M$. Starting from the first two lines of (6.3) and making use of (2.36e), we get the following elliptic equations with source terms on the right

$$\begin{aligned} &2\mathring{\eth}\mathring{\eth}'\hat{\varphi}_{-2}^{(0)} - 4\hat{\varphi}_{-2}^{(0)} \\ &= -2a\mathcal{L}_\eta\mathcal{L}_\xi\hat{\varphi}_{-2}^{(0)} + 2M\mathcal{L}_\xi\hat{\varphi}_{-2}^{(1)} + \frac{2Ma}{a^2+r^2}\mathcal{L}_\eta\hat{\varphi}_{-2}^{(1)} + \frac{6ar}{a^2+r^2}\mathcal{L}_\eta\hat{\varphi}_{-2}^{(0)} - 2MV\hat{\varphi}_{-2}^{(1)} \\ &\quad - \tfrac{1}{4}\big(4a^2 + 9(\kappa_1 - \bar{\kappa}_{1'})^2\big)\mathcal{L}_\xi\mathcal{L}_\xi\hat{\varphi}_{-2}^{(0)} - 6(\kappa_1 - \bar{\kappa}_{1'})\mathcal{L}_\xi\hat{\varphi}_{-2}^{(0)} - \frac{3(a^4 + a^2r^2 - 2Mr^3)\hat{\varphi}_{-2}^{(0)}}{(a^2+r^2)^2} \\ &\quad - \frac{2M(Ma^2 + a^2r - 3Mr^2 + r^3)\hat{\varphi}_{-2}^{(1)}}{(a^2+r^2)^2}, \end{aligned} \tag{6.43a}$$

$$\begin{aligned} &2\mathring{\eth}\mathring{\eth}'\hat{\varphi}_{-2}^{(1)} - 6\hat{\varphi}_{-2}^{(1)} \\ &= -2a\mathcal{L}_\eta\mathcal{L}_\xi\hat{\varphi}_{-2}^{(1)} + 2M\mathcal{L}_\xi\hat{\varphi}_{-2}^{(2)} + \frac{2Ma}{a^2+r^2}\mathcal{L}_\eta\hat{\varphi}_{-2}^{(2)} + \frac{2ar}{a^2+r^2}\mathcal{L}_\eta\hat{\varphi}_{-2}^{(1)} + \frac{6a(a^2-r^2)}{M(a^2+r^2)}\mathcal{L}_\eta\hat{\varphi}_{-2}^{(0)} \\ &\quad - 2MV\hat{\varphi}_{-2}^{(2)} - \tfrac{1}{4}\big(4a^2 + 9(\kappa_1 - \bar{\kappa}_{1'})^2\big)\mathcal{L}_\xi\mathcal{L}_\xi\hat{\varphi}_{-2}^{(1)} - 6(\kappa_1 - \bar{\kappa}_{1'})\mathcal{L}_\xi\hat{\varphi}_{-2}^{(1)} \end{aligned}$$

$$- \frac{6r(-a^4 - 3Ma^2r - a^2r^2 + Mr^3)\hat{\varphi}^{(0)}_{-2}}{M(a^2+r^2)^2} - \frac{(7a^4 - 20Ma^2r + 7a^2r^2 + 6Mr^3)\hat{\varphi}^{(1)}_{-2}}{(a^2+r^2)^2}. \quad (6.43b)$$

It is then manifest that

$$\begin{aligned}
&(2\mathring{\eth}\mathring{\eth}' - 4)\hat{\varphi}^{(0)}_{-2} \\
&\quad = O_\infty(r^{-2})M^2\mathcal{L}_\eta\hat{\varphi}^{(1)}_{-2} + O_\infty(r^{-1})MrV\hat{\varphi}^{(1)}_{-2} + O_\infty(r^{-1})M\hat{\varphi}^{(1)}_{-2} + O_\infty(r^{-1})M\mathcal{L}_\eta\hat{\varphi}^{(0)}_{-2} \\
&\qquad + O_\infty(r^{-1})M\hat{\varphi}^{(0)}_{-2} + O_\infty(1)M\mathcal{L}_\eta\mathcal{L}_\xi\hat{\varphi}^{(0)}_{-2} + O_\infty(1)M^2\mathcal{L}_\xi\mathcal{L}_\xi\hat{\varphi}^{(0)}_{-2} + O_\infty(1)M\mathcal{L}_\xi\hat{\varphi}^{(0)}_{-2} \\
&\qquad + O_\infty(1)M\mathcal{L}_\xi\hat{\varphi}^{(1)}_{-2}, \qquad (6.44a) \\
&(2\mathring{\eth}\mathring{\eth}' - 6)\hat{\varphi}^{(1)}_{-2} \\
&\quad = O_\infty(r^{-2})M^2\mathcal{L}_\eta\hat{\varphi}^{(2)}_{-2} + O_\infty(r^{-1})MrV\hat{\varphi}^{(2)}_{-2} + O_\infty(r^{-1})M\mathcal{L}_\eta\hat{\varphi}^{(1)}_{-2} + O_\infty(r^{-1})M\hat{\varphi}^{(1)}_{-2} \\
&\qquad + O_\infty(1)M\mathcal{L}_\eta\mathcal{L}_\xi\hat{\varphi}^{(1)}_{-2} + O_\infty(1)M^2\mathcal{L}_\xi\mathcal{L}_\xi\hat{\varphi}^{(1)}_{-2} + O_\infty(1)M\mathcal{L}_\xi\hat{\varphi}^{(1)}_{-2} + O_\infty(1)M\mathcal{L}_\xi\hat{\varphi}^{(2)}_{-2} \\
&\qquad + O_\infty(1)\hat{\varphi}^{(0)}_{-2} + O_\infty(1)\mathcal{L}_\eta\hat{\varphi}^{(0)}_{-2}, \qquad (6.44b)
\end{aligned}$$

and commuting with rV gives

$$\begin{aligned}
&(2\mathring{\eth}\mathring{\eth}' - 4)rV\hat{\varphi}^{(0)}_{-2} \\
&\quad = O_\infty(r^{-2})M^2\mathcal{L}_\eta rV\hat{\varphi}^{(1)}_{-2} + O_\infty(r^{-2})M^2\mathcal{L}_\eta\hat{\varphi}^{(1)}_{-2} + O_\infty(r^{-1})MrV(rV\hat{\varphi}^{(1)}_{-2}) + O_\infty(r^{-1})MrV\hat{\varphi}^{(1)}_{-2} \\
&\qquad + O_\infty(r^{-1})MrV\hat{\varphi}^{(1)}_{-2} + O_\infty(r^{-1})M\hat{\varphi}^{(1)}_{-2} + O_\infty(r^{-1})M\mathcal{L}_\eta rV\hat{\varphi}^{(0)}_{-2} \\
&\qquad + O_\infty(r^{-1})M\mathcal{L}_\eta\hat{\varphi}^{(0)}_{-2} + O_\infty(r^{-1})MrV\hat{\varphi}^{(0)}_{-2} + O_\infty(r^{-1})M\hat{\varphi}^{(0)}_{-2} \\
&\qquad + O_\infty(1)M\mathcal{L}_\eta\mathcal{L}_\xi rV\hat{\varphi}^{(0)}_{-2} + O_\infty(1)M^2\mathcal{L}_\xi\mathcal{L}_\xi rV\hat{\varphi}^{(0)}_{-2} + O_\infty(1)M\mathcal{L}_\xi rV\hat{\varphi}^{(0)}_{-2} + O_\infty(1)M\mathcal{L}_\xi rV\hat{\varphi}^{(1)}_{-2}, \\
&\qquad\qquad (6.45a) \\
&(2\mathring{\eth}\mathring{\eth}' - 6)rV\hat{\varphi}^{(1)}_{-2} \\
&\quad = O_\infty(r^{-1})MrV(rV\hat{\varphi}^{(2)}_{-2}) + O_\infty(r^{-1})MrV\hat{\varphi}^{(2)}_{-2} + O_\infty(r^{-2})M^2\mathcal{L}_\eta rV\hat{\varphi}^{(2)}_{-2} + O_\infty(r^{-2})M^2\mathcal{L}_\eta\hat{\varphi}^{(2)}_{-2} \\
&\qquad + O_\infty(r^{-1})M\mathcal{L}_\eta rV\hat{\varphi}^{(1)}_{-2} + O_\infty(r^{-1})M\mathcal{L}_\eta\hat{\varphi}^{(1)}_{-2} + O_\infty(r^{-1})MrV\hat{\varphi}^{(1)}_{-2} + O_\infty(r^{-1})M\hat{\varphi}^{(1)}_{-2} \\
&\qquad + O_\infty(1)M\mathcal{L}_\eta\mathcal{L}_\xi rV\hat{\varphi}^{(1)}_{-2} + O_\infty(1)M^2\mathcal{L}_\xi\mathcal{L}_\xi rV\hat{\varphi}^{(1)}_{-2} + O_\infty(1)M\mathcal{L}_\xi rV\hat{\varphi}^{(1)}_{-2} + O_\infty(1)M\mathcal{L}_\xi rV\hat{\varphi}^{(2)}_{-2} \\
&\qquad + O_\infty(1)rV\hat{\varphi}^{(0)}_{-2} + O_\infty(r^{-1})M\hat{\varphi}^{(0)}_{-2} + O_\infty(1)\mathcal{L}_\eta rV\hat{\varphi}^{(0)}_{-2} + O_\infty(r^{-1})M\mathcal{L}_\eta\hat{\varphi}^{(0)}_{-2}. \qquad (6.45b)
\end{aligned}$$

For $r \geq 10M$, the left-hand side of each subequation in both (6.44) and (6.45) is a strongly elliptic operator (with its maximal eigenvalue uniformly bounded away from zero) acting the field. On the right-hand sides of both (6.44a) and (6.45a), the source terms involving $\mathcal{L}_\xi$ derivatives have better $t^{-1+\delta}$ pointwise decay, and when obtaining pointwise, energy, and Morawetz estimates for the terms on the right-hand side, r inverse coefficients will give t inverse decay since $r \geq t$ in the exterior region. Hence we apply an elliptic estimate to (6.44a), and this together with the pointwise estimate (6.28) yields

$$|\mathcal{L}_\xi^j\hat{\psi}_{-2}|_{k-17-3j,\mathbb{D}} \lesssim rv^{-1}t^{-(1-\delta)(7/2+j)+\delta}(\mathbb{I}^k_{-2})^{1/2}. \quad (6.46)$$

Here, the nonzero j cases come from the fact that $\mathcal{L}_\xi$ is a symmetry of the systems (6.44) and (6.45). We can also obtain an energy and Morawetz estimate for $\delta \leq \alpha \leq 2-\delta$ from the energy and Morawetz estimate (6.27) that

$$\begin{aligned}
&\|\mathcal{L}_\xi^j\hat{\psi}_{-2}\|^2_{W^{k-10-3j}_{-2}(\Sigma^{\text{ext}}_t)} + \|r\mathcal{L}_\xi^jV\hat{\psi}_{-2}\|^2_{W^{k-11-3j}_{\alpha-2}(\Sigma^{\text{ext}}_t)} + \|\mathcal{L}_\xi^j\hat{\psi}_{-2}\|^2_{W^{k-11-3j}_{\alpha-3}(\Omega^{\text{ext}}_{t,\infty})} \\
&\quad \lesssim t^{-(1-\delta)(8+2j)+\alpha-\delta}\mathbb{I}^k_{-2}. \qquad (6.47)
\end{aligned}$$

Substituting these two estimates into (6.44b) and (6.45b), and applying again elliptic estimates, this yields improved exterior estimates for $\mathcal{L}_\xi^j\hat{\psi}_{-2}^{(1)}$ for $\delta\leq\alpha\leq 2-\delta$:

$$|\mathcal{L}_\xi^j\hat{\psi}_{-2}^{(1)}|_{k-18-3j,\mathbb{D}}\lesssim rv^{-1}t^{-(1-\delta)(7/2+j)+\delta}(\mathbb{I}_{-2}^k)^{1/2}, \tag{6.48a}$$

$$\|\mathcal{L}_\xi^j\hat{\psi}_{-2}^{(1)}\|^2_{W_{-2}^{k-11-3j}(\Sigma_t^{\text{ext}})}+\|r\mathcal{L}_\xi^jV\hat{\psi}_{-2}^{(1)}\|^2_{W_{\alpha-2}^{k-12-3j}(\Sigma_t^{\text{ext}})}+\|\mathcal{L}_\xi^j\hat{\psi}_{-2}^{(1)}\|^2_{W_{\alpha-3}^{k-12-3j}(\Omega_{t,\infty}^{\text{ext}})}$$
$$\lesssim t^{-(1-\delta)(8+2j)+\alpha-\delta}\mathbb{I}_{-2}^k. \tag{6.48b}$$

The above two estimates together prove the $i=1$ case of the estimates (6.38) and (6.39). From the preliminary estimates (6.46) and (6.47) for $\mathcal{L}_\xi^j\hat{\psi}_{-2}$, from estimates (6.48a) and (6.48b) for $\mathcal{L}_\xi^j\hat{\psi}_{-2}^{(1)}$, from equations (6.44a) and (6.45a), and from elliptic estimates, there are the following improved estimates for $\mathcal{L}_\xi^j\hat{\psi}_{-2}$

$$|\mathcal{L}_\xi^j\hat{\psi}_{-2}|_{k-21-3j,\mathbb{D}}\lesssim rv^{-1}t^{-(1-\delta)(9/2+j)+\delta}(\mathbb{I}_{-2}^k)^{1/2}, \tag{6.49a}$$

$$\|\mathcal{L}_\xi^j\hat{\psi}_{-2}\|^2_{W_{-2}^{k-14-3j}(\Sigma_t^{\text{ext}})}+\|r\mathcal{L}_\xi^jV\hat{\psi}_{-2}\|^2_{W_{\alpha-2}^{k-15-3j}(\Sigma_t^{\text{ext}})}+\|\mathcal{L}_\xi^j\hat{\psi}_{-2}\|^2_{W_{\alpha-3}^{k-15-3j}(\Omega_{t,\infty}^{\text{ext}})}$$
$$\lesssim t^{-(1-\delta)(10+2j)+\alpha-\delta}\mathbb{I}_{-2}^k, \tag{6.49b}$$

which is the $i=0$ case of (6.38) and (6.39).

Let us turn to point (2) now. The estimates (6.40) and (6.41) are proved in lemma 6.12, so we consider only the estimates (6.42). We note that these estimates only improve the r decay compared to the pointwise estimate (6.28) and Morawetz estimate (6.27), hence in the following proof we will restrict to $r\geq 10M$ region where the left-hand sides of (6.44) are both strongly elliptic operators acting on the field.

From the pointwise estimate (6.28) and Morawetz estimate (6.27), an elliptic estimate applied to (6.44a) gives that

$$|\mathcal{L}_\xi^j\hat{\psi}_{-2}|_{k-17-3j,\mathbb{D}}\lesssim r^\delta v^{-1}t^{-(1-\delta)(\frac{5}{2}+j)+\delta}(\mathbb{I}_{-2}^k)^{1/2}, \tag{6.50a}$$

$$\|\mathcal{L}_\xi^j\hat{\psi}_{-2}\|^2_{W_{-1-\delta}^{k-11-3j}(\Omega_{t,\infty}^{\text{int}})}\lesssim t^{-(1-\delta)(6+2j)}\mathbb{I}_{-2}^k. \tag{6.50b}$$

Turning to (6.44b), we make use of these estimates of $\mathcal{L}_\xi^j\hat{\psi}_{-2}$, the pointwise estimate (6.28), and Morawetz estimate (6.27), and obtain from elliptic estimates that

$$|\mathcal{L}_\xi^j\hat{\psi}_{-2}^{(1)}|_{k-18-3j,\mathbb{D}}\lesssim r^\delta v^{-1}t^{-(1-\delta)(\frac{5}{2}+j)+\delta}(\mathbb{I}_{-2}^k)^{1/2}, \tag{6.51a}$$

$$\|\mathcal{L}_\xi^j\hat{\psi}_{-2}^{(1)}\|^2_{W_{-1-\delta}^{k-12-3j}(\Omega_{t,\infty}^{\text{int}})}\lesssim t^{-(1-\delta)(6+2j)}\mathbb{I}_{-2}^k. \tag{6.51b}$$

Notice that the first estimate is exactly the estimate (6.42c). From the estimate (6.51b), it follows that for any $l\in\mathbb{N}$ and $0\leq\alpha<(6+2j)(1-\delta)$,

$$\|\mathcal{L}_\xi^j\hat{\psi}_{-2}^{(1)}\|^2_{W_{\alpha-1-\delta}^{k-12-3j}(\Omega_{2^lt,2^{l+1}t}^{\text{int}})}\lesssim (2^lt)^{-(6+2j)(1-\delta)+\alpha}\mathbb{I}_{-2}^k. \tag{6.52}$$

Summing over these estimates from $l=0$ to ∞, this proves (6.42d).

In the same manner, we obtain the preliminary estimate for $\hat{\psi}_{-2}$ that, for $0\leq\alpha<(6+2j)(1-\delta)$,

$$\|\mathcal{L}_\xi^j\hat{\psi}_{-2}\|^2_{W_{\alpha-1-\delta}^{k-11-3j}(\Omega_{t,\infty}^{\text{int}})}\lesssim t^{-(6+2j)(1-\delta)+\alpha}\mathbb{I}_{-2}^k. \tag{6.53}$$

Substituting the pointwise estimates (6.50a) and (6.42c) and Morawetz estimates (6.53) and (6.42d) back in to (6.44a), we conclude from elliptic estimates that estimates (6.42a) and (6.42b) hold. □

7. The spin-weight +2 Teukolsky equation

In this section, we consider the field $\hat{\psi}_{+2}$ that satisfies equation (3.25b).

7.1. **Basic assumptions.** Let us first introduce two different basic energy and Morawetz (BEAM) conditions and one pointwise condition.

Definition 7.1 (BEAM conditions and pointwise condition for $\hat{\psi}_{+2}$)**.** Let $\hat{\psi}_{+2}$ be a spin-weight $+2$ scalar that is a solution of the Teukolsky equation (3.25b). For a spin-weighted scalar φ and $k \in \mathbb{Z}^+$, let the energies $E^k_{\Sigma_t}(\varphi)$ and $E^k_{\Sigma_{\text{init}}}(\varphi)$ be defined as in definition 6.6, and the spacetime integral $B^k_{t_1,t_2}[\varphi]$ be defined as in definition 6.7. Two BEAM conditions and one pointwise condition are defined to be that

(1) (First BEAM condition) for all sufficiently large $k \in \mathbb{N}$ and any $t_2 \geq t_1 \geq t_0$,

$$\sum_{i=0}^{2}\left(E^k_{\Sigma_{t_2}}(M^{4-i}(r^2Y)^i(r^{-4}\hat{\psi}_{+2})) + B^k_{t_1,t_2}(M^{4-i}(r^2Y)^i(r^{-4}\hat{\psi}_{+2}))\right) \lesssim \sum_{i=0}^{2}E^k_{\Sigma_{t_1}}(M^{4-i}(r^2Y)^i(r^{-4}\hat{\psi}_{+2})). \tag{7.1}$$

(2) (Second BEAM condition) there is a $\delta_0 \in (0, 1/2)$ such that for all sufficiently large $k \in \mathbb{N}$ and any $t_2 \geq t_1 \geq t_0$,

$$\begin{aligned}&\sum_{i=0}^{1}\left(E^k_{\Sigma_{t_2}}(M^{i+\frac{\delta_0}{2}}r^{-\frac{\delta_0}{2}}Y^i\hat{\psi}_{+2}) + B^k_{t_1,t_2}(M^{i+\frac{\delta_0}{2}}r^{-\frac{\delta_0}{2}}Y^i\hat{\psi}_{+2})\right)\\ &+ E^k_{\Sigma_{t_2}}(M^2Y^2\hat{\psi}_{+2}) + B^k_{t_1,t_2}(M^2Y^2\hat{\psi}_{+2})\\ &\quad\lesssim \sum_{i=0}^{1}E^k_{\Sigma_{t_1}}(M^{i+\frac{\delta_0}{2}}r^{-\frac{\delta_0}{2}}Y^i\hat{\psi}_{+2}) + E^k_{\Sigma_{t_1}}(M^2Y^2\hat{\psi}_{+2}).\end{aligned} \tag{7.2}$$

(3) (Pointwise condition) for all sufficiently large $k \in \mathbb{N}$,

$$\lim_{t\to\pm\infty}\left(|\hat{\psi}_{+2}|_{k,\not{D}}\big|_{\mathscr{I}^+}\right) \to 0. \tag{7.3}$$

The pointwise condition (3) in definition 7.1 is one of the basic assumptions used in section 8, and either of the two BEAM conditions in the above definition together with the assumption that $\mathbb{I}^{+2,k}_{\text{init}}$ is bounded are shown in theorem 7.8 to imply this pointwise condition.

Remark 7.2. Compared to the quantities introduced by Ma in [33, Appendix A], which are denoted here by $\hat{\phi}^{i,\text{Ma}}_{+2}$, we have $\hat{\psi}_{+2} = \frac{1}{4}(a^2+r^2)^{5/2}\kappa_1{}^2\bar{\kappa}_{1'}{}^{-2}\hat{\phi}^{0,\text{Ma}}_{+2}$ where the first factor $\frac{1}{4}(a^2+r^2)^{5/2}$ is to make the quantity nondegenerate at future null infinity, and the other factor $\kappa_1{}^2\bar{\kappa}_{1'}{}^{-2}$ corresponds to a spin rotation of the frame. The quantities $\hat{\phi}^{i,\text{Ma}}_{+2}$ $(i = 0, 1, 2)$ and the quantity $\hat{\psi}_{+2}$ are related by

$$r\hat{\phi}^{i,\text{Ma}}_{+2} = \sum_{j=0}^{i}O_\infty(1)(M^{-1}r^2Y)^j(M^4r^{-4}\hat{\psi}_{+2}). \tag{7.4}$$

As a preliminary, the following relations between the two BEAM conditions are useful.

Lemma 7.3. *Let $0 < \delta_0 < 1/2$ be fixed. The BEAM condition* (1) *in definition 7.1 implies BEAM condition* (2) *in definition 7.1.*

Proof. For ease of presentation we will here use mass normalization as in definition 4.4. The lemma follows from adapting the proof of [33, Proposition 3.1.2] to our hyperboloidal foliation. By arguing in the same way as in the proof of [33, Proposition 3.1.2] except that the integration is over $\Omega^{R_0-M}_{t_1,t_2}$, and using the relation (7.4), one finds that there exists a constant $R_0 \geq 10M$

such that for any $k \geq 1$,

$$\begin{aligned}
&\sum_{i=0}^{1}\left(E^k_{\Sigma^{R_0}_{t_2}}(r^{4-2i-\frac{\delta_0}{2}}(r^2Y)^i(r^{-4}\hat{\psi}_{+2})) + \int_{\Omega^{R_0}_{t_1,t_2}} r^{-3}\sum_{|\mathbf{a}|\leq k}|\mathbb{B}^{\mathbf{a}}(r^{4-2i}(r^2Y)^i(r^{-4}\hat{\psi}_{+2}))|^2 \mathrm{d}^4\mu\right)\\
&\quad\lesssim \sum_{i=0}^{1}\left(E^k_{\Sigma^{R_0-M}_{t_1}}(r^{4-2i-\frac{\delta_0}{2}}(r^2Y)^i(r^{-4}\hat{\psi}_{+2})) + E^k_{\Sigma^{R_0-M,R_0}_{t_2}}(r^{4-2i-\frac{\delta_0}{2}}(r^2Y)^i(r^{-4}\hat{\psi}_{+2}))\right)\\
&\qquad + \int_{\Omega^{R_0-M,R_0}_{t_1,t_2}} r^{-1}\sum_{|\mathbf{a}|\leq k}|\mathbb{B}^{\mathbf{a}}(r^{4-2i}(r^2Y)^i(r^{-4}\hat{\psi}_{+2}))|^2 \mathrm{d}^4\mu. \qquad (7.5)
\end{aligned}$$

The $k > 1$ case here follows from commuting the Killing symmetry $\mathcal{L}_\xi$ (which is timelike for $r \geq R_0 - M \geq 9M$) and elliptic estimates. Combining the BEAM condition (1) with the above estimate (7.5), and from the following facts

$$r^{4-\frac{\delta_0}{2}}(r^{-4}\hat{\psi}_{+2}) = O_\infty(1)r^{-\frac{\delta_0}{2}}\hat{\psi}_{+2}, \qquad (7.6a)$$

$$r^{2-\frac{\delta_0}{2}}(r^2Y)(r^{-4}\hat{\psi}_{+2}) = O_\infty(1)r^{-\frac{\delta_0}{2}}Y\hat{\psi}_{+2} + O_\infty(r^{-1})r^{-\frac{\delta_0}{2}}\hat{\psi}_{+2}, \qquad (7.6b)$$

$$(r^2Y)^2(r^{-4}\hat{\psi}_{+2}) = O_\infty(1)Y^2\hat{\psi}^{(0)}_{+2} + O_\infty(r^{-1+\frac{\delta_0}{2}})r^{-\frac{\delta_0}{2}}Y\hat{\psi}_{+2} + O_\infty(r^{-2+\frac{\delta_0}{2}})r^{-\frac{\delta_0}{2}}\hat{\psi}_{+2}, \qquad (7.6c)$$

the estimate (7.2) is valid. □

7.2. **The estimates.** This section uses the r^p lemma 5.6 to obtain decay estimates for $\hat{\psi}_{+2}$. One can perform a rescaling to $\hat{\psi}_{+2}$ as follows such that the governing equation of the new scalar can be put into the form of (5.29) with $\vartheta = 0$, to which the r^p lemma 5.6 can be applied.

Lemma 7.4. *Given a spin-weight +2 scalar $\hat{\psi}_{+2}$ that satisfies equation (3.25b), the quantity $\hat{\varphi}^{(0)}_{+2}$ defined by*

$$\hat{\varphi}^{(0)}_{+2} = \frac{(a^2+r^2)^2\hat{\psi}_{+2}}{\Delta^2} \qquad (7.7)$$

then satisfies

$$\begin{aligned}
\widehat{\mathbb{S}}_{+2}(\hat{\varphi}^{(0)}_{+2}) = {}& \frac{8ar}{a^2+r^2}\mathcal{L}_\eta\hat{\varphi}^{(0)}_{+2} - \frac{8(Ma^2+a^2r-3Mr^2+r^3)}{\Delta}V\hat{\varphi}^{(0)}_{+2}\\
&+\frac{4r(9Ma^2+a^2r-7Mr^2+r^3)\hat{\varphi}^{(0)}_{+2}}{(a^2+r^2)^2}. \qquad (7.8)
\end{aligned}$$

Before proving weighted r^p estimates for (7.8), we state some equivalent relations between the energy norms of $\hat{\psi}_{+2}$ and $\hat{\varphi}^{(0)}_{+2}$, which turn out to be useful in translating r^p estimates of $\hat{\varphi}^{(0)}_{+2}$ to r^p estimates of $\hat{\psi}_{+2}$.

Lemma 7.5. *Let $\hat{\psi}_{+2}$ be a spin-weight +2 scalar. Let $\hat{\varphi}^{(0)}_{+2}$ be as in equation (7.7). Let $k \in \mathbb{N}$, $\beta \in \mathbb{R}$ and $R_0 \geq 10M$. There is the bound*

$$\|\hat{\varphi}^{(0)}_{+2}\|_{W^k_\beta(\Sigma^{R_0}_t)} \sim \|\hat{\psi}_{+2}\|_{W^k_\beta(\Sigma^{R_0}_t)}. \qquad (7.9)$$

Furthermore, for $\alpha \in [0,2]$ and $k \geq 1$,

$$\|rV\hat{\varphi}^{(0)}_{+2}\|_{W^{k-1}_{\alpha-2}(\Sigma^{R_0}_t)} + \|\hat{\varphi}^{(0)}_{+2}\|_{W^k_{-2}(\Sigma^{R_0}_t)} \sim \|rV\hat{\psi}_{+2}\|_{W^{k-1}_{\alpha-2}(\Sigma^{R_0}_t)} + \|\hat{\psi}_{+2}\|_{W^k_{-2}(\Sigma^{R_0}_t)}. \qquad (7.10)$$

Proof. These estimates follow easily by arguing in the same way as in lemma 6.5 and taking into account the relation (7.7). □

Now we are ready to apply the r^p lemma 5.6 to equation (7.8) and to state the α-weighted estimate, which is a combination of the r^p estimate for $\hat{\varphi}^{(0)}_{+2}$ and the BEAM estimate (2) in definition 7.1.

Lemma 7.6 (r^p estimate for $\hat{\psi}_{+2}$)**.** *Let $\hat{\psi}_{+2}$ be a spin-weight $+2$ scalar that is a solution of the Teukolsky equation (3.25b). Assume either of the BEAM conditions from definition 7.1 holds. Then, for all sufficiently large $k \in \mathbb{N}$, any $0 < \delta \leq \delta_0$, $\alpha \in [\delta, 2-\delta]$ and $t_2 \geq t_1 \geq t_0$,*

$$\begin{aligned}
&\|\hat{\psi}_{+2}\|^2_{W^{k+1}_{-2}(\Sigma_{t_2})} + \|rV\hat{\psi}_{+2}\|^2_{W^k_{\alpha-2}(\Sigma_{t_2})} + E^{k+1}_{\Sigma_{t_2}}(M^{1+\frac{\delta_0}{2}} r^{-\frac{\delta_0}{2}} Y\hat{\psi}_{+2}) \\
&\quad + E^{k+1}_{\Sigma_{t_2}}(M^2 Y^2 \hat{\psi}_{+2}) + \|\hat{\psi}_{+2}\|^2_{W^k_{\alpha-3}(\Omega_{t_1,t_2})} \\
&\lesssim \|\hat{\psi}_{+2}\|^2_{W^{k+1}_{-2}(\Sigma_{t_1})} + \|rV\hat{\psi}_{+2}\|^2_{W^k_{\alpha-2}(\Sigma_{t_1})} + E^{k+1}_{\Sigma_{t_1}}(M^{1+\frac{\delta_0}{2}} r^{-\frac{\delta_0}{2}} Y\hat{\psi}_{+2}) + E^{k+1}_{\Sigma_{t_1}}(M^2 Y^2 \hat{\psi}_{+2}).
\end{aligned} \tag{7.11}$$

Proof. From lemma 7.3, we only need to prove this lemma under the assumption that BEAM condition (2) from definition 7.1 is satisfied. In the following, we assume that such an assumption holds.

By putting equation (7.8) into the form of (5.29), we see that $\vartheta = 0$ and the assumptions in lemma 5.6 are satisfied with

$$b_{V,-1} = 8 > 0, \qquad b_\phi = MO_\infty(r^{-1}), \qquad b_{0,0} + 2 + 2 = 0, \tag{7.12}$$

and the spin weight is $+2$. Thus, we apply the r^p lemma 5.6 and obtain that for any $k \in \mathbb{N}$, $t_0 \leq t_1 \leq t_2$, $0 < \delta \leq \delta_0$ and $\alpha \in [\delta, 2-\delta]$, there are constants $R_0 = R_0(k) \geq 10M$ and $C = C(k)$ such that

$$\begin{aligned}
&\|rV\hat{\varphi}^{(0)}_{+2}\|^2_{W^k_{\alpha-2}(\Sigma^{R_0}_{t_2})} + \|\hat{\varphi}^{(0)}_{+2}\|^2_{W^{k+1}_{-2}(\Sigma^{R_0}_{t_2})} \\
&+ \|\mathring{\eth}' \hat{\varphi}^{(0)}_{+2}\|^2_{W^k_{\alpha-3}(\Omega^{R_0}_{t_1,t_2})} + \|\hat{\varphi}^{(0)}_{+2}\|^2_{W^k_{\alpha-3}(\Omega^{R_0}_{t_1,t_2})} \\
&\leq C\Bigg(\|rV\hat{\varphi}^{(0)}_{+2}\|^2_{W^k_{\alpha-2}(\Sigma^{R_0}_{t_1})} + \|\hat{\varphi}^{(0)}_{+2}\|^2_{W^{k+1}_{-2}(\Sigma^{R_0}_{t_1})} \\
&\qquad + \|\hat{\varphi}^{(0)}_{+2}\|^2_{W^{k+1}_0(\Omega^{R_0-M,R_0}_{t_1,t_2})} + \sum_{t\in\{t_1,t_2\}} \|\hat{\varphi}^{(0)}_{+2}\|^2_{W^1_\alpha(\Sigma^{R_0-M,R_0}_t)}\Bigg).
\end{aligned} \tag{7.13}$$

This is an r^p estimate for $\hat{\varphi}^{(0)}_{+2}$. From lemma 7.5, $\hat{\varphi}^{(0)}_{+2}$ can be replaced by $\hat{\psi}_{+2}$ in this estimate. By adding this r^p estimate of $\hat{\psi}_{+2}$ to the assumed BEAM estimate (7.2), the estimate (7.11) follows. □

Lemma 7.7. *Under the same assumptions of lemma 7.6, the estimate (7.11) holds as well if we replace the right-hand side by $\mathbb{I}^{k+3;1}_{\text{init}}(\hat{\psi}_{+2})$ as in definition 4.20.*

Proof. For ease of presentation we will here use mass normalization as in definition 4.4. To prove this result, we just need to show the following estimate which bounds the norms on Σ_{t_0} by those on Σ_{init}:

$$\begin{aligned}
&\|\hat{\psi}_{+2}\|^2_{W^{k+1}_{-2}(\Sigma_{t_0})} + \|rV\hat{\psi}_{+2}\|^2_{W^k_{\alpha-2}(\Sigma_{t_0})} \\
&+ E^{k+1}_{\Sigma_{t_0}}(r^{-\frac{\delta_0}{2}} Y\hat{\psi}_{+2}) + E^{k+1}_{\Sigma_{t_0}}(Y^2\hat{\psi}_{+2}) \lesssim \mathbb{I}^{k+3;1}_{\text{init}}(\hat{\psi}_{+2}).
\end{aligned} \tag{7.14}$$

Applying lemma 5.7 to the spin-weighted wave equation (7.8) in the early region, and from the relation between $\hat{\varphi}^{(0)}_{+2}$ and $\hat{\psi}_{+2}$ norms in lemma 7.5, it follows that for $\alpha \in [\delta, 2-\delta]$, there is a constant $R_0 = R_0(k) \geq 10M$ such that

$$\begin{aligned}
&\|rV\hat{\psi}_{+2}\|^2_{W^k_{\alpha-2}(\Sigma^{R_0}_{t_0})} + \|\hat{\psi}_{+2}\|^2_{W^{k+1}_{-2}(\Sigma^{R_0}_{t_0})} + \|\hat{\psi}_{+2}\|^2_{W^{k+1}_{\alpha-3}(\Omega^{\text{early},R_0}_{\text{init},t_0})} \\
&\lesssim \|\hat{\psi}_{+2}\|_{H^{k+1}_{\alpha-1}(\Sigma_{\text{init}})} + \|\hat{\psi}_{+2}\|^2_{W^{k+1}_0(\Omega^{\text{early},R_0-M,R_0}_{\text{init},t_0})} + \|\hat{\psi}_{+2}\|^2_{W^{k+1}_{-\delta}(\Sigma^{R_0-M,R_0}_{t_0})}.
\end{aligned} \tag{7.15}$$

Since R_0 is bounded, $\|\hat{\psi}_{+2}\|^2_{W^{k+1}_0(\Omega^{\text{early},R_0-M,R_0}_{\text{init},t_0})}$ and $\|\hat{\psi}_{+2}\|^2_{W^{k+1}_{-\delta}(\Sigma^{R_0-M,R_0}_{t_0})}$ are both bounded by a multiple of an initial norm $\mathbb{I}^{k+1;1}_{\text{init}}(\hat{\psi}_{+2})$, by standard exponential growth estimates for wave-like equations. For the same reason, the sum of $\|rV\hat{\psi}_{+2}\|^2_{W^k_{\alpha-2}(\Sigma^{r_+,R_0}_{t_0})}$ and $\|\hat{\psi}_{+2}\|^2_{W^{k+1}_{-2}(\Sigma^{r_+,R_0}_{t_0})}$ is bounded by $\mathbb{I}^{k+1;1}_{\text{init}}(\hat\psi_{+2})$ as well. For the first term on the right of (7.15), since $\alpha \le 2-\delta$, it holds that

$$\int_{\Sigma_{\text{init}}} \sum_{|\mathbf{a}|\le k+1} r^{\alpha+2|\mathbf{a}|-2}|\mathbb{B}^{\mathbf{a}}\hat\psi_{+2}|^2 \mathrm{d}^3\mu \le \int_{\Sigma_{\text{init}}} \sum_{|\mathbf{a}|\le k+1} r^{-\delta+2|\mathbf{a}|}|\mathbb{B}^{\mathbf{a}}\hat\psi_{+2}|^2 \mathrm{d}^3\mu \lesssim \mathbb{I}^{k+1;1}_{\text{init}}(\hat\psi_{+2}). \tag{7.16}$$

Thus, for any $\alpha \in [\delta, 2-\delta]$,

$$\|rV\hat\psi_{+2}\|^2_{W^k_{\alpha-2}(\Sigma_{t_0})} + \|\hat\psi_{+2}\|^2_{W^{k+1}_{-2}(\Sigma_{t_0})} \lesssim \mathbb{I}^{k+1;1}_{\text{init}}(\hat\psi_{+2}). \tag{7.17}$$

In addition, since MY belongs to the operator set $\mathbb{D}$,

$$E^{k+1}_{\Sigma_{t_0}}(r^{-\frac{\delta_0}{2}}Y\hat\psi_{+2}) + E^{k+1}_{\Sigma_{t_0}}(Y^2\hat\psi_{+2}) \lesssim \|rV\hat\psi_{+2}\|^2_{W^{k+2}_{-\delta}(\Sigma_{t_0})} + \|\hat\psi_{+2}\|^2_{W^{k+3}_{-2}(\Sigma_{t_0})} \lesssim \mathbb{I}^{k+3;1}_{\text{init}}(\hat\psi_{+2}), \tag{7.18}$$

where the second step follows from (7.17). The above two estimates together imply the inequality (7.14), which then completes the proof. □

Theorem 7.8 (Decay estimates for $\mathcal{L}^j_\xi\hat\psi_{+2}$)**.** *Let $\hat\psi_{+2}$ be a spin-weight $+2$ scalar that is a solution of the Teukolsky equation (3.25b). Assume either of the BEAM conditions from definition 7.1 holds. Assume furthermore that $\mathbb{I}^{k;1}_{\text{init}}(\hat\psi_{+2})$ is finite for all $k \in \mathbb{N}$. Under these conditions:*

(1) the pointwise condition (3) in definition 7.1 holds;

(2) furthermore, there is a regularity constant K such that for all $j \in \mathbb{N}$, sufficiently large $k-K-3j$, $0<\delta\le\delta_0$, $\delta\le\alpha\le 2-\delta$, and $t\ge t_0$, there are the energy and Morawetz estimates

$$\|\mathcal{L}^j_\xi\hat\psi_{+2}\|^2_{W^{k-K-7j}_{-2}(\Sigma_t)} + \|rV\mathcal{L}^j_\xi\hat\psi_{+2}\|^2_{W^{k-K-1-7j}_{\alpha-2}(\Sigma_t)} + \|\mathcal{L}^j_\xi\hat\psi_{+2}\|^2_{W^{k-K-1-7j}_{\alpha}(\Omega_{t,\infty})} \lesssim t^{\alpha-2+\delta-(2-2\delta)j}\mathbb{I}^{k;1}_{\text{init}}(\hat\psi_{+2}), \tag{7.19}$$

and pointwise decay estimates

$$|\mathcal{L}^j_\xi\hat\psi_{+2}|_{k-K-7j,\mathbb{D}} \lesssim rv^{-1}t^{-(1-\delta)(\frac{1}{2}+j)+\delta}(\mathbb{I}^{k;1}_{\text{init}}(\hat\psi_{+2}))^{1/2}. \tag{7.20}$$

Remark 7.9. The pointwise condition (3) in definition 7.1 is the main result in this section which is used in section 8. This theorem implies that the assumption of the pointwise condition (3) from definition 7.1 can be replaced by an assumption of either of the two BEAM conditions from definition 7.1 together with the assumption that $\mathbb{I}^{k;1}_{\text{init}}(\hat\psi_{+2})$ is finite for any $k\in\mathbb{N}$.

Proof. For ease of presentation we will here use mass normalization as in definition 4.4.

First, consider the limits along $\mathscr{I}^+$ as $t\to\pm\infty$. Let $r(t)$ denote the value of r corresponding to the intersection of Σ_{init} and Σ_t. For R_0 fixed and t sufficiently negative, we have that $r(t) > R_0$ and $r(t)\sim -t$. Recall that the proof of the r^p lemma 5.7 is based on an application of Stokes' theorem, so we may replace Σ_{init} by $\Sigma_{\text{init}}\cap\{r>r(t)\}$. The region under consideration is $r>R_0$, so we may drop all the terms supported on $r\in[R_0-M,R_0]$, which gives

$$\|rV\hat\psi_{+2}\|^2_{W^k_{-1}(\Sigma_t)} + \|\hat\psi_{+2}\|^2_{W^{k+1}_{-2}(\Sigma_t)} \lesssim \|\hat\psi_{+2}\|^2_{H^{k+1}_0(\Sigma_{\text{init}}\cap\{r>r(t)\})}. \tag{7.21}$$

From adapting the proof of the Sobolev lemma 4.32 on Σ_t, in particular from estimate (4.50), one finds

$$\lim_{r\to\infty}\int_{S^2}|\hat\psi_{+2}(t,r,\omega)|^2_k\mathrm{d}^2\mu \lesssim \left(\|rV\hat\psi_{+2}\|^2_{W^k_{-1}(\Sigma_{t_2})} + \|\hat\psi_{+2}\|^2_{W^{k+1}_{-2}(\Sigma_{t_2})}\right) + \int_{S^2}|\hat\psi_{+2}(t,r(t),\omega)|^2_k\mathrm{d}^2\mu. \tag{7.22}$$

Adapting the bound on $\hat\psi_{+2}$ on Σ_{init} in lemma 4.36 and applying the previous estimate on the energy on Σ_t, one finds

$$|\hat\psi_{+2}|^2_{k,\not{\mathbb{D}}}\big|_{\mathscr{I}^+} = \lim_{r\to\infty}\int_{S^2}|\hat\psi_{+2}(t,r,\omega)|^2_k\mathrm{d}^2\mu \lesssim \|\hat\psi_{+2}\|^2_{H^{k+1}_0(\Sigma_{\text{init}}\cap\{r>r(t)\})}, \tag{7.23}$$

which goes to zero as $t\to-\infty$. (In fact, this argument gives a rate, but we do not need to calculate the rate for the pointwise condition (3).) As $t\to\infty$, the pointwise decay estimates

(7.20) implies that $\lim_{t\to\infty}\left(|\hat{\psi}_{+2}|_{k,\eth}\big|_{\mathscr{I}^+}\right)\to 0$ holds for any $k\in\mathbb{N}$, and hence the first claim holds.

Based on the above discussion and from lemma 7.6, to prove this theorem, we only need to show the second claim under the assumption that the conclusions of lemma 7.6 are valid. For a general spin-weighted scalar φ, define

$$\widetilde{F}(\varphi,k,\alpha,t)=\begin{cases}\|\varphi\|^2_{W^{k-2}_{-2}(\Sigma_t)}+\|rV\varphi\|^2_{W^{k-3}_{\alpha-2}(\Sigma_t)}+E^{k-2}_{\Sigma_t}(r^{-\frac{\delta_0}{2}}Y\varphi)+E^{k-2}_{\Sigma_t}(Y^2\varphi) & \text{if } \alpha\in[\delta,2-\delta]\\ 0 & \text{if } \alpha<\delta\end{cases}\tag{7.24a}$$

$$\widetilde{G}(\varphi,k,\alpha,t)=\|\varphi\|^2_{W^k_{\alpha-2}(\Sigma_t)}.\tag{7.24b}$$

We shall prove that the energy and Morawetz estimates (7.19) follows if we can show

$$\begin{aligned}&\widetilde{F}(\mathcal{L}^j_\xi\hat{\psi}_{+2},k-6-7j,\alpha,t)+\int_t^\infty\widetilde{G}(\mathcal{L}^j_\xi\hat{\psi}_{+2},k-9-7j,\alpha-1,t')\mathrm{d}t'\\&\lesssim t^{\alpha-2+\delta-(2-2\delta)j}\mathbb{I}^{k;1}_{\text{init}}(\hat{\psi}_{+2}).\end{aligned}\tag{7.25}$$

Estimate (7.11) in lemma 7.6 can be stated as, for $\alpha\in[\delta,2-\delta]$,

$$\widetilde{F}(\hat{\psi}_{+2},k+3,\alpha,t_2)+\int_{t_1}^{t_2}\widetilde{G}(\hat{\psi}_{+2},k,\alpha-1,t)\mathrm{d}t\lesssim\widetilde{F}(\hat{\psi}_{+2},k+3,\alpha,t_1),\tag{7.26}$$

and note from (7.24) that

$$\widetilde{G}(\hat{\psi}_{+2},k,\alpha-1,t)\gtrsim\widetilde{F}(\hat{\psi}_{+2},k,\alpha-1,t),\tag{7.27}$$

hence for any $k_1\le k\in\mathbb{N}$, $\delta\le\alpha\le 2-\delta$, and $t_0\le t_1\le t_2$,

$$\widetilde{F}(\hat{\psi}_{+2},k+3,\alpha,t_2)+\int_{t_1}^{t_2}\widetilde{F}(\hat{\psi}_{+2},k,\alpha-1,t)\mathrm{d}t\lesssim\widetilde{F}(\hat{\psi}_{+2},k+3,\alpha,t_1).\tag{7.28}$$

This can be put into the form of (5.6d) by taking $D=\gamma=0$ and performing the following replacement

$$\delta\mapsto\alpha_1,\qquad 2-\delta\mapsto\alpha_2,\qquad \widetilde{F}(\hat{\psi}_{+2},k+3,\alpha,t)\mapsto F(\lfloor\tfrac{k+3}{3}\rfloor,\alpha,t).\tag{7.29}$$

An application of lemma 5.2 then yields for $\alpha\in[\delta,2-\delta]$,

$$\widetilde{F}(\hat{\psi}_{+2},k-6,\alpha,t)+\int_t^\infty\widetilde{G}(\hat{\psi}_{+2},k-9,\alpha-1,t')\mathrm{d}t'\lesssim t^{\alpha-2+\delta}\widetilde{F}(\hat{\psi}_{+2},k,2-\delta,t_0).\tag{7.30}$$

From lemma 7.7 (or estimate (7.14)), it holds that

$$\widetilde{F}(\hat{\psi}_{+2},k,2-\delta,t_0)\lesssim\mathbb{I}^{k;1}_{\text{init}}(\hat{\psi}_{+2}),\tag{7.31}$$

hence this proves the $j=0$ case of (7.25).

We prove the general j case of (7.25) by induction. Assume that estimate (7.25) holds for $j=j'$, so that

$$\widetilde{F}(\mathcal{L}^{j'}_\xi\hat{\psi}_{+2},k-6-7j',\delta,t)\lesssim t^{-2+2\delta-(2-2\delta)j'}\mathbb{I}^{k;1}_{\text{init}}(\hat{\psi}_{+2}).\tag{7.32}$$

Since $\mathcal{L}_\xi$ is a symmetry of (3.25b), it holds that for any $j\in\mathbb{N}$, k sufficiently large, δ sufficiently small, $\alpha\in[\delta,2-\delta]$, and $t_2\ge t_1\ge t_0$,

$$\widetilde{F}(\mathcal{L}^j_\xi\hat{\psi}_{+2},k+3,\alpha,t_2)+\int_{t_1}^{t_2}\widetilde{G}(\mathcal{L}^j_\xi\hat{\psi}_{+2},k,\alpha-1,t)\mathrm{d}t\lesssim\widetilde{F}(\mathcal{L}^j_\xi\hat{\psi}_{+2},k+3,\alpha,t_1).\tag{7.33}$$

One can argue similarly to the proof of lemma 6.12 to obtain better decay estimates for $\mathcal{L}^j_\xi\hat{\psi}_{+2}$ as follows. Rescaling equation (2.36e) for $\widehat{\boxed{\mathrm{S}}}_{+2}$, we can isolate the term $r^2V\mathcal{L}_\xi\hat{\psi}_{+2}$ from (3.25b) and write $r^2V\mathcal{L}_\xi\hat{\psi}_{+2}$ as a weighted sum of $(rV)^2\hat{\psi}_{+2}$, $rV\hat{\psi}_{+2}$, $r^{-1}\mathcal{L}_\eta(rV\hat{\psi}_{+2})$, $r^{-1}\mathcal{L}_\eta\hat{\psi}_{+2}$, $S_s\hat{\psi}_{+2}$, $\mathcal{L}_\xi\hat{\psi}_{+2}$, and $\hat{\psi}_{+2}$ all with $O_\infty(1)$ coefficients. Therefore,

$$\|rV\mathcal{L}^{j'+1}_\xi\hat{\psi}_{+2}\|^2_{W^{k-2}_{-\delta}(\Sigma_t)}\lesssim\|r^2V\mathcal{L}^{j'+1}_\xi\hat{\psi}_{+2}\|^2_{W^{k-2}_{-\delta-2}(\Sigma_t)}\lesssim\|\mathcal{L}^{j'}_\xi\hat{\psi}_{+2}\|^2_{W^k_{-2}(\Sigma_t)},\tag{7.34}$$

which furthermore implies

$$\widetilde{F}(\mathcal{L}_\xi^{j'+1}\hat{\psi}_{+2}, k-7-7j', 2-\delta, t) \lesssim \widetilde{F}(\mathcal{L}_\xi^{j'}\hat{\psi}_{+2}, k-6-7j', \delta, t) \lesssim t^{-2+2\delta-(2-2\delta)j'}\mathbb{I}^{k;1}_{\text{init}}(\hat{\psi}_{+2}). \tag{7.35}$$

A repeated application of lemma 5.2 as above to (7.33) but with $j \mapsto j'+1$ and $k \mapsto k-10-7j'$ then yields

$$\begin{aligned}&\widetilde{F}(\mathcal{L}_\xi^{j'+1}\hat{\psi}_{+2}, k-7-7j'-6, \alpha, t) + \int_t^\infty \widetilde{G}(\mathcal{L}_\xi^{j'+1}\hat{\psi}_{+2}, k-7-7j'-9, \alpha-1, t')\mathrm{d}t' \\ &\quad \lesssim t^{\alpha-2+\delta-2+2\delta-(2-2\delta)j'}\mathbb{I}^{k;1}_{\text{init}}(\hat{\psi}_{+2}).\end{aligned} \tag{7.36}$$

This proves the $j = j'+1$ case of (7.25), which completes the induction and justifies the estimate (7.25) for general $j \in \mathbb{N}$ cases and hence the estimate (7.19).

As to the pointwise decay estimates, the proof is the same as the one for lemma 6.12. From the Sobolev inequality (4.49) with $\gamma = \delta$ and the energy estimate (7.25) with $\alpha = 1+\delta$ and $\alpha = 1-\delta$, one finds

$$\begin{aligned}|\mathcal{L}_\xi^j\hat{\psi}_{+2}|^2_{k-11-7j,\mathbb{D}} &\lesssim \left(\widetilde{F}(\mathcal{L}_\xi^j\hat{\psi}_{+2}, k-6-7j, 1+\delta, t)\widetilde{F}(\mathcal{L}_\xi^j\hat{\psi}_{+2}, k-6-7j, 1-\delta, t)\right)^{\frac{1}{2}} \\ &\lesssim t^{-(1-\delta)(1+2j)}\mathbb{I}^{k;1}_{\text{init}}(\hat{\psi}_{+2}).\end{aligned} \tag{7.37}$$

Alternatively, having already established the limit as $t \to \infty$ is zero, one can now apply the anisotropic spacetime Sobolev inequality (4.53) and the Morawetz estimate (7.25) with $\alpha = \delta$ to obtain

$$\begin{aligned}|\mathcal{L}_\xi^j r^{-1}\hat{\psi}_{+2}|^2_{k-19-7j,\mathbb{D}} &\lesssim \|\mathcal{L}_\xi^j r^{-1}\hat{\psi}_{+2}\|^{1/2}_{W^{k-16-7j}_{-1}(\Omega_{t,\infty})}\|\mathcal{L}_\xi^{j+1} r^{-1}\hat{\psi}_{+2}\|^{1/2}_{W^{k-16-7j}_{-1}(\Omega_{t,\infty})} \\ &\lesssim \|\mathcal{L}_\xi^j\hat{\psi}_{+2}\|^{1/2}_{W^{k-16-7j}_{-3+\delta}(\Omega_{t,\infty})}\|\mathcal{L}_\xi^{j+1}\hat{\psi}_{+2}\|^{1/2}_{W^{k-9-7(j+1)}_{-3+\delta}(\Omega_{t,\infty})} \\ &\lesssim t^{-(1-\delta)(3+2j)}\mathbb{I}^{k;1}_{\text{init}}(\hat{\psi}_{+2}).\end{aligned} \tag{7.38}$$

Combining the two pointwise estimates and observing $v^{-1} \lesssim \min(r^{-1}, t^{-1})$ give the desired pointwise decay estimate (7.20). ∎

8. The metric and core connection coefficients

We shall now use the results presented in Sections 6 and 7 to prove pointwise, energy, and Morawetz estimates for linearized gravity from the transport form of the equations of linearized gravity in ORG gauge, derived in section 3.3. We shall work in terms of the compactified hyperboloidal coordinate system (t, R, θ, ϕ) where t is the hyperboloidal time introduced in section 2.4, $R = 1/r$, and θ, ϕ are the angular coordinates in the ingoing Eddington-Finkelstein coordinate system. We shall sometimes use the notation $\omega = (\theta, \phi)$. In terms of this coordinate system, future null infinity $\mathscr{I}^+$ is located at $R = 0$. For our considerations here, we may without loss of generality consider compactly supported initial data, in which case the solution of the Teukolsky equation is smooth at $\mathscr{I}^+$ in the compactified hyperboloidal coordinate system, cf. section 4.2.

Definition 8.1. A set of linearized Einstein fields is defined to consist of the following:

(1) a linearized metric δg_{ab},
(2) linearized metric components $G_{0i'}$ from section 3.1,
(3) linearized connection and connection coefficients from section 3.1,
(4) linearized curvature components from (1.12),
(5) rescaled linearized curvature components $\hat{\psi}_{-2}$ and $\hat{\psi}_{+2}$ from definition 3.5, and
(6) the core quantities $\hat{\sigma}'$, $\widehat{G}_2$, $\hat{\tau}'$, $\widehat{G}_1$, $\hat{\beta}'$, and $\widehat{G}_0$ from definition 3.7.

Definition 8.2. An outgoing BEAM solution of the linearized Einstein equation is defined to be be a set of linearized Einstein fields as in definition 8.1 such that

(1) δg_{ab} satisfies the linearized Einstein equation (1.3) in the outgoing radiation gauge (1.5),
(2) $\hat{\psi}_{-2}$ satisfies the BEAM condition from definition 6.8,
(3) $\hat{\psi}_{+2}$ satisfies the pointwise decay condition, point 3 of definition 7.1.

8.1. **Expansions at infinity and transport equations.**

Definition 8.3. Let f be a spin-weighted scalar on $\mathscr{I}^+$ which decays sufficiently rapidly at i_0, and define

$$(\mathbf{I}f)(t,\omega) = \int_{-\infty}^{t} f(t',\omega)\mathrm{d}t'. \tag{8.1}$$

For a non-negative integer i, define $\mathbf{I}^i$ by

$$\mathbf{I}^i = \mathbf{I}\circ\mathbf{I}^{i-1}, \tag{8.2}$$

with $\mathbf{I}^0$ the identity operator.

It is now possible to define an expansion at null infinity. This depends on a level of regularity k, an order of the expansion l, an order m up to which the expansion terms vanish, a weight parameter α_1, and a positive constant D^2. In the case that $m = l+1$, then all the terms in the expansion vanish, and the scalar is estimated solely by the remainder term.

Definition 8.4 ((k,l,m,α_1,D^2) expansion)**.** Let $k,l,m\in\mathbb{N}$ be such that $0\le m\le l+1$. Let $\alpha_1\in\mathbb{R}$. Let $D>0$.

In the exterior region where $r\ge t$, a spin-weighted scalar φ is defined to have a $(k,l,0,\alpha_1,D^2)$ expansion if, for $i\in\{0,\ldots,l\}$, there are functions φ_i on $\mathscr{I}^+$ and there is a function $\varphi_{\text{rem};l}$ in the exterior such that

$$\forall(t,r,\omega): \qquad \varphi(t,r,\omega) = \sum_{i=0}^{l}\frac{R^i}{i!}\varphi_i(t,\omega)+\varphi_{\text{rem};l}(t,r,\omega), \tag{8.3a}$$

$$\|\varphi_{\text{rem};l}\|^2_{W^k_{\alpha_1-3}(\Omega^{\text{ext}}_{t_0,\infty})}\lesssim D^2, \tag{8.3b}$$

$$\|\varphi_{\text{rem};l}\|^2_{W^k_{\alpha_1-3}(\Omega^{\text{early}}_{\text{init},t_0})}\lesssim D^2, \tag{8.3c}$$

$$\forall t\in\mathbb{R},\forall i\in\{0,\ldots,l\}: \qquad \int_{S^2}|\varphi_i(t,\omega)|^2_{k,\eth}\mathrm{d}^2\mu\lesssim D^2\langle t\rangle^{2i-\alpha_1+1}, \tag{8.3d}$$

$$\forall\omega\in S^2, 0\le i<j\le l+1, |\mathbf{a}|\le k: \qquad \lim_{t\to\infty}(\mathbf{I}^{j-i}\eth^{\mathbf{a}}\varphi_i)(t,\omega)=0. \tag{8.3e}$$

If, furthermore, for $m\in\mathbb{Z}^+$, the expansion terms up to order $m-1\ge 0$ vanish, i.e.

$$\forall t\in\mathbb{R},\forall i\in\{0,\ldots,m-1\}: \qquad \varphi_i(t,\omega)=0, \tag{8.4}$$

then we say φ has a (k,l,m,α_1,D^2) expansion.

Because $Yt = h'(r)$, when trying to solve $Y\varphi=\varrho$ in terms of expansions from null infinity, one finds that the expansion coefficients for φ are coupled through the expansion coefficients in $h'(r)$. The following lemma handles this coupling.

Lemma 8.5. *Given any $l\in\mathbb{N}$, for $k\in\{0,\ldots,l\}$, define a_k and $b_k(R)$ to be such that*

$$\frac{1}{h'(r)} = \sum_{k=0}^{l} a_k R^k + b_l(R)R^{l+1}, \tag{8.5}$$

and define $b_{-1}(R) = 1/h'(r)$.

Let ϱ and φ be spin-weighted scalars. Let φ_{init} be a spin-weighted scalar on Σ_{init}.

If φ solves

$$Y\varphi=\varrho, \tag{8.6a}$$

$$\varphi|_{\Sigma_{\text{init}}}=\varphi_{\text{init}} \tag{8.6b}$$

with ϱ having the expansion

$$\varrho=\sum_{i=0}^{j}\frac{R^i}{i!}\varrho_i(t,\omega)+\varrho_{\text{rem};j}, \tag{8.7}$$

then φ is given by

$$\varphi=\sum_{i=0}^{j-1}\frac{R^i}{i!}\varphi_i(t,\omega)+\varphi_{\text{rem};j-1}, \tag{8.8}$$

where

$$\varphi_i(t,\omega) = \sum_{k=0}^{i} \frac{i!a_{i-k}}{k!}\tilde{\varphi}_k(t,\omega), \qquad 0 \le i \le j-1, \tag{8.9a}$$

$$\varphi_{\mathrm{rem};j-1} = \sum_{i=0}^{j} \frac{b_{j-i-1}(R)}{i!} R^j \tilde{\varphi}_i(t,\omega) + \tilde{\varphi}_{\mathrm{rem};j} + \tilde{\varphi}_{\mathrm{hom};j}, \tag{8.9b}$$

$$\tilde{\varphi}_0(t,\omega) = \mathbf{I}\varrho_0(t,\omega), \tag{8.9c}$$

$$\tilde{\varphi}_i(t,\omega) = \mathbf{I}\big(\varrho_i(t,\omega) - \sum_{k=0}^{i-1} \frac{(i-1)!i!a_{i-k-1}}{k!}\tilde{\varphi}_k(t,\omega)\big), \qquad 1 \le i \le j, \tag{8.9d}$$

$\tilde{\varphi}_{\mathrm{rem};j}$ *is the solution of*

$$Y\tilde{\varphi}_{\mathrm{rem};j} = \varrho_{\mathrm{rem};j} - \sum_{i=0}^{j} \frac{1}{i!}(b'_{j-i-1}(R)R + jb_{j-i-1}(R))R^{j+1}\tilde{\varphi}_i(t,\omega), \tag{8.10}$$

$$\tilde{\varphi}_{\mathrm{rem};j}|_{\Sigma_{\mathrm{init}}} = 0, \tag{8.11}$$

and $\tilde{\varphi}_{\mathrm{hom};j}$ *is the solution of*

$$Y\tilde{\varphi}_{\mathrm{hom};j} = 0, \tag{8.12}$$

$$\tilde{\varphi}_{\mathrm{hom};j}(t_{\mathrm{init}}(r),r,\omega) = \varphi_{\mathrm{init}}(r,\omega) - \sum_{i=0}^{j} \frac{R^i}{i!h'(r)}\tilde{\varphi}_i(t_{\mathrm{init}}(r),\omega), \tag{8.13}$$

where $t_{\mathrm{init}}(r) = t_0 - h(r)/2$ *is the value of* t *on* Σ_{init} *at* r.

Proof. Make an ansatz

$$\varphi = \sum_{i=0}^{j} \frac{R^i}{i!h'(r)}\tilde{\varphi}_i(t,\omega) + \tilde{\varphi}_{\mathrm{rem};j} + \tilde{\varphi}_{\mathrm{hom};j}. \tag{8.14}$$

This gives

$$Y\varphi = \sum_{i=0}^{j}\left(Y\Big(\frac{R^i}{i!h'(r)}\Big)\tilde{\varphi}_i(t,\omega) + \frac{R^i}{i!}\partial_t\tilde{\varphi}_i(t,\omega)\right) + Y\tilde{\varphi}_{\mathrm{rem};j} + Y\tilde{\varphi}_{\mathrm{hom};j}. \tag{8.15}$$

We set $l = j-i-1$ in (8.5) and calculate

$$Y\left(\frac{R^i}{h'(r)}\right) = \sum_{k=0}^{l} a_k(i+k)R^{i+k+1} + (b'_l(R)R + (i+l+1)b_l(R))R^{i+l+2}. \tag{8.16}$$

Substituting this into (8.15) gives

$$\begin{aligned} Y\varphi ={}& \sum_{i=0}^{j} \frac{R^i}{i!}\partial_t\tilde{\varphi}_i(t,\omega) + \sum_{i=0}^{j}\sum_{k=0}^{j-i-1} \frac{a_k(i+k)}{i!}R^{i+k+1}\tilde{\varphi}_i(t,\omega) \\ &+ \sum_{i=0}^{j} \frac{1}{i!}(b'_{j-i-1}(R)R + jb_{j-i-1}(R))R^{j+1}\tilde{\varphi}_i(t,\omega) + Y\tilde{\varphi}_{\mathrm{rem};j} + Y\tilde{\varphi}_{\mathrm{hom};j} \\ ={}& \sum_{i=0}^{j} \frac{R^i}{i!}\partial_t\tilde{\varphi}_i(t,\omega) + \sum_{i=1}^{j}\sum_{k=0}^{i-1} \frac{a_{i-k-1}(i-1)}{k!}R^i\tilde{\varphi}_k(t,\omega) \\ &+ \sum_{i=0}^{j} \frac{1}{i!}(b'_{j-i-1}(R)R + jb_{j-i-1}(R))R^{j+1}\tilde{\varphi}_i(t,\omega) + Y\tilde{\varphi}_{\mathrm{rem};j} + Y\tilde{\varphi}_{\mathrm{hom};j}. \end{aligned} \tag{8.17}$$

If one now imposes conditions (8.9c) and (8.9d) on the $\tilde{\varphi}_i$, then

$$\varrho_0(t,\omega) = \partial_t\tilde{\varphi}_0(t,\omega), \tag{8.18a}$$

$$\varrho_i(t,\omega) = \partial_t\tilde{\varphi}_i(t,\omega) + \sum_{k=0}^{i-1} \frac{a_{i-k-1}(i-1)i!}{k!}\tilde{\varphi}_k(t,\omega), \qquad 1 \le i \le j. \tag{8.18b}$$

If one further imposes that $\tilde{\varphi}_{\mathrm{rem};l}$ and $\tilde{\varphi}_{\mathrm{hom};j}$ satisfy the differential equations (8.10) and (8.12) respectively, then equation (8.17) becomes $Y\varphi = \varrho$. If one imposes the initial conditions (8.11) on $\tilde{\varphi}_{\mathrm{rem};j}$ and (8.13) on $\tilde{\varphi}_{\mathrm{hom};j}$, then one finds that φ satisfies the initial condition $\varphi|_{\Sigma_{\mathrm{init}}} = \varphi_{\mathrm{init}}$.

Now applying the expansion (8.5) for $(h')^{-1}$ with $l = j - i - 1$ in equation (8.14), gathering like powers of R, and putting the R^l term with the remainder term, one finds that

$$\begin{aligned}\varphi &= \sum_{i=0}^{j}\sum_{k=0}^{j-i-1}\frac{a_k}{i!}R^{i+k}\tilde{\varphi}_i(t,\omega) + \sum_{i=0}^{j}\frac{b_{j-i-1}(R)}{i!}R^j\tilde{\varphi}_i(t,\omega) + \tilde{\varphi}_{\mathrm{rem};j} + \tilde{\varphi}_{\mathrm{hom};j}\\ &= \sum_{i=0}^{j-1}\sum_{k=0}^{i}\frac{a_{i-k}}{k!}R^i\tilde{\varphi}_k(t,\omega) + \sum_{i=0}^{j}\frac{b_{j-i-1}(R)}{i!}R^j\tilde{\varphi}_i(t,\omega) + \tilde{\varphi}_{\mathrm{rem};j} + \tilde{\varphi}_{\mathrm{hom};j}. \end{aligned}\tag{8.19}$$

By comparing this expansion with the expansion (8.8), we finally get (8.9a) and (8.9b). □

Lemma 8.6 (Propagation of expansions)**.** *Let $\delta > 0$ be sufficiently small. Let φ and ϱ be spin-weighted scalars, and let φ_{init} be a spin-weighted scalar on Σ_{init}. Let $k[\varrho], l[\varrho], m[\varrho] \in \mathbb{N}$, $\alpha_1[\varrho] > 0$, and $D[\varrho] > 0$ be such that $l[\varrho] \geq 1$ and $2l[\varrho] + 3 + \delta \leq \alpha_1[\varrho] \leq 2l[\varrho] + 4 - \delta$. If φ solves*

$$Y\varphi = \varrho, \tag{8.20a}$$

$$\varphi|_{\Sigma_{\mathrm{init}}} = \varphi_{\mathrm{init}} \tag{8.20b}$$

and ϱ has a $(k[\varrho], l[\varrho], m[\varrho], \alpha_1[\varrho], D[\varrho]^2)$ expansion, then the following hold:

(1) With

$$k[\varphi] = k[\varrho], \tag{8.21a}$$

$$l[\varphi] = l[\varrho] - 1, \tag{8.21b}$$

$$m[\varphi] = \min(m[\varrho], l[\varphi] + 1), \tag{8.21c}$$

$$\alpha_1[\varphi] = \alpha_1[\varrho] - 2 - \delta, \tag{8.21d}$$

$$D[\varphi]^2 = D[\varrho]^2 + \mathbb{I}^{k[\varphi]+1;2l[\varphi]+3}_{\mathrm{init}}(\varphi), \tag{8.21e}$$

φ has a $(k[\varphi], l[\varphi], m[\varphi], \alpha_1[\varphi], D[\varphi]^2)$ expansion.

(2) For any $q \in \{0, 1\}$, and $t \geq t_0$, φ satisfies

$$\|\mathcal{L}_\xi^q\varphi\|^2_{W^{k[\varphi]-q}_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})} \lesssim_{l[\varphi]} D[\varphi]^2 t^{-2q} \tag{8.22}$$

and in the exterior region where $r \geq t$ that

$$\text{for } m[\varrho] \leq l[\varrho], \qquad |\varphi|^2_{k[\varphi]-3,\mathbb{D}} \lesssim_{l[\varphi]} D[\varphi]^2 r^{-2m[\varrho]}t^{-\alpha_1[\varphi]+1+2m[\varrho]}, \tag{8.23a}$$

$$\text{for } m[\varrho] = l[\varrho] + 1, \qquad |\varphi|^2_{k[\varphi]-3,\mathbb{D}} \lesssim_{l[\varphi]} D[\varphi]^2 r^{-\alpha_1[\varphi]+1}. \tag{8.23b}$$

Proof. For ease of presentation, throughout this proof, we use mass normalization as in definition 4.4 and use $\lesssim$ to mean $\lesssim_{l[\varphi]}$. Since, by assumption, ϱ has an expansion, one can apply lemma 8.5 to obtain an expansion for φ. In the following, for simplicity, we use k to denote $k[\varphi] = k[\varrho]$.

Step 1: Treat the $\tilde{\varphi}_i$. We first show in this step that

$$\forall t \in \mathbb{R}, q \in \{0,1\}, i \in \{0,\dots,l[\varrho]\}, \qquad \int_{S^2}|\mathcal{L}_\xi^q\tilde{\varphi}_i(t,\omega)|^2_{k-q,\not{\mathbb{D}}}\mathrm{d}^2\mu \lesssim D[\varrho]^2\langle t\rangle^{2i-\alpha_1[\varrho]+3-2q}, \tag{8.24a}$$

$$\forall \omega \in S^2, 0 \leq i < j \leq l[\varrho], |\mathbf{a}| \leq k, \qquad \lim_{t\to\infty}(\mathbf{I}^{j-i}\not{\mathbb{D}}^{\mathbf{a}}\tilde{\varphi}_i)(t,\omega) = 0, \tag{8.24b}$$

and

$$\forall t \in \mathbb{R}, \forall i \in \{0,\dots,m[\varrho]-1\}, \qquad \tilde{\varphi}_i = 0. \tag{8.25}$$

From (8.9c) and (8.9d), it is clear that (8.25) holds, and hence (8.24) holds true for $i \in \{0,\dots,m[\varrho]-1\}$. Furthermore, if $m[\varrho] = l[\varrho]+1$, all the $\{\tilde{\varphi}_i\}_{i=0}^{l[\varrho]}$ vanish, and (8.24) is manifestly valid. Hence, we only need to prove (8.24) below for $m[\varrho] \leq i \leq l[\varrho]$.

The remaining $m[\varrho] \leq i \leq l[\varrho]$ cases are treated by induction. First, consider the $i = m[\varrho]$ case. Since $\alpha_1[\varrho] > 2l[\varrho] + 3 \geq 2l[\varphi] + 3$, the expression (8.9d) for $\tilde{\varphi}_i$ and the integrability and decay conditions for $\varrho_{m[\varrho]}$ give, for any $t \geq t_0$,

$$\not{D}^{\mathbf{a}}\tilde{\varphi}_{m[\varrho]}(t,\omega) = \int_{-\infty}^{t} \not{D}^{\mathbf{a}}\varrho_{m[\varrho]}(t',\omega)\mathrm{d}t' = -\int_{t}^{\infty} \not{D}^{\mathbf{a}}\varrho_{m[\varrho]}(t',\omega)\mathrm{d}t', \tag{8.26a}$$

$$\not{D}^{\mathbf{a}}\mathcal{L}_\xi\tilde{\varphi}_{m[\varrho]}(t,\omega) = \not{D}^{\mathbf{a}}\varrho_{m[\varrho]}(t,\omega), \tag{8.26b}$$

$$\lim_{t\to\infty}(\mathbf{I}^j\not{D}^{\mathbf{a}}\tilde{\varphi}_{m[\varrho]})(t,\omega) = \lim_{t\to\infty}(\mathbf{I}^{j+1}\not{D}^{\mathbf{a}}\varrho_{m[\varrho]})(t,\omega) = 0, \qquad 0 \leq j \leq l[\varrho]-m[\varrho], |\mathbf{a}| \leq k, \tag{8.26c}$$

and, for any $t \geq t_0$,

$$\begin{aligned}\int_{S^2}|\mathcal{L}_\xi\tilde{\varphi}_{m[\varrho]}(t,\omega)|^2_{k,\not{D}}\mathrm{d}^2\mu &\leq \int_{S^2}|\varrho_{m[\varrho]}(t,\omega)|^2_{k,\not{D}}\mathrm{d}^2\mu \\ &\lesssim D[\varrho]^2 t^{-\alpha_1[\varrho]+1+2m[\varrho]},\end{aligned} \tag{8.27}$$

$$\begin{aligned}\int_{S^2}|\tilde{\varphi}_{m[\varrho]}(t,\omega)|^2_{k,\not{D}}\mathrm{d}^2\mu &\leq \int_{S^2}\left(\int_t^\infty|\varrho_{m[\varrho]}(t',\omega)|_{k,\not{D}}\mathrm{d}t'\right)^2\mathrm{d}^2\mu \\ &\leq \left(\int_t^\infty\left(\int_{S^2}|\varrho_{m[\varrho]}(t',\omega)|^2_{k,\not{D}}\mathrm{d}^2\mu\right)^{1/2}\mathrm{d}t'\right)^2 \\ &\lesssim \left(D[\varrho]\int_t^\infty(t')^{-\alpha_1[\varrho]/2+1/2+m[\varrho]}\mathrm{d}t'\right)^2 \\ &\lesssim D[\varrho]^2t^{-\alpha_1[\varrho]+3+2m[\varrho]}\end{aligned} \tag{8.28}$$

where the second step of (8.28) follows from Minkowski's integral inequality. Similarly, for $t \leq -t_0$ and $q \in \{0,1\}$, one has $\int_{S^2}|\mathcal{L}^q_\xi\tilde{\varphi}_{m[\varrho]}(t,\omega)|^2_{k-q,\not{D}}\mathrm{d}^2\mu \lesssim D[\varrho]^2|t|^{-\alpha_1[\varrho]+3+2m[\varrho]-2q}$, and, for $t \in [-t_0,t_0]$, one has that $\int_{S^2}|\mathcal{L}^q_\xi\tilde{\varphi}_{m[\varrho]}(t,\omega)|_{k-q,\not{D}}\mathrm{d}^2\mu$ is bounded. These prove the $i = m[\varrho]$ case of (8.24).

For induction, let $l' \leq l[\varrho]$, and suppose that the estimates (8.24) hold for $m[\varrho] \leq i \leq l'-1$. From the expression (8.9d) for the $\tilde{\varphi}_i$, the decay and integrability conditions for ϱ_i, the assumption that $\alpha_1[\varrho] > 2l[\varrho]+3$, and the inductive hypothesis, one finds that, for any $m[\varrho] \leq i \leq l' \leq l[\varrho]$, $q \in \{0,1\}$, and $t \geq t_0$,

$$\begin{aligned}\not{D}^{\mathbf{a}}\tilde{\varphi}_i(t,\omega) &= \int_{-\infty}^t\left(\not{D}^{\mathbf{a}}\varrho_i(t',\omega) - \sum_{j=0}^{i-1}\frac{a_{i-j-1}(i-1)i!}{j!}\not{D}^{\mathbf{a}}\tilde{\varphi}_j(t',\omega)\right)\mathrm{d}t' \\ &= \int_t^\infty\left(\not{D}^{\mathbf{a}}\varrho_i(t',\omega) - \sum_{j=0}^{i-1}\frac{a_{i-j-1}(i-1)i!}{j!}\not{D}^{\mathbf{a}}\tilde{\varphi}_j(t',\omega)\right)\mathrm{d}t',\end{aligned} \tag{8.29}$$

$$\begin{aligned}&\int_{S^2}|\mathcal{L}^q_\xi\tilde{\varphi}_i(t,\omega)|^2_{k-q,\not{D}}\mathrm{d}^2\mu \\ &\quad\leq \int_{S^2}\left(\int_t^\infty\left(|\mathcal{L}^q_\xi\varrho_i(t',\omega)|_{k-q,\not{D}} + \sum_{j=0}^{i-1}\frac{a_{i-j-1}(i-1)i!}{j!}|\mathcal{L}^q_\xi\tilde{\varphi}_j(t',\omega)|_{k-q,\not{D}}\right)\mathrm{d}t'\right)^2\mathrm{d}^2\mu \\ &\quad\lesssim \left(\int_t^\infty\left(\int_{S^2}\left(|\mathcal{L}^q_\xi\varrho_i(t',\omega)|^2_{k-q,\not{D}} + \sum_{j=0}^{i-1}|\mathcal{L}^q_\xi\tilde{\varphi}_j(t',\omega)|^2_{k-q,\not{D}}\right)\mathrm{d}^2\mu\right)^{1/2}\mathrm{d}t'\right)^2 \\ &\quad\lesssim \left(D[\varrho]\int_t^\infty\left((t')^{i-\alpha_1[\varrho]/2+1/2} + \sum_{j=0}^{i-1}(t')^{j-\alpha_1[\varrho]/2+3/2-q}\right)\mathrm{d}t'\right)^2 \\ &\quad\lesssim D[\varrho]^2t^{2i-\alpha_1[\varrho]+3-2q}.\end{aligned} \tag{8.30}$$

Similarly, for $t \leq -t_0$, one finds $\int_{S^2}|\mathcal{L}^q_\xi\tilde{\varphi}_i(t,\omega)|^2_{k-q,\not{D}}\mathrm{d}^2\mu \lesssim D[\varrho]^2|t|^{2i-\alpha_1[\varrho]+3-2q}$, and, for $t \in [-t_0,t_0]$, one has that $\int_{S^2}|\mathcal{L}^q_\xi\tilde{\varphi}_i(t,\omega)|^2_{k-q,\not{D}}\mathrm{d}^2\mu$ is bounded. These together imply

$$\forall t\in\mathbb{R}, \forall i \in \{m[\varrho],\dots,l'\}, \forall q \in \{0,1\}: \qquad \int_{S^2}|\mathcal{L}^q_\xi\tilde{\varphi}_i(t,\omega)|^2_{k-q,\not{D}}\mathrm{d}^2\mu \lesssim D[\varrho]^2\langle t\rangle^{2i-\alpha_1[\varrho]+3-2q}. \tag{8.31}$$

Furthermore, for i satisfying $m[\varrho] \leq i \leq j \leq l' \leq l[\varrho]$, one finds

$$\lim_{t\to\infty}(\mathbf{I}^{j-i}\not{\mathbb{D}}^{\mathbf{a}}\tilde{\varphi}_i)(t,\omega) = \lim_{t\to\infty}(\mathbf{I}^{j-i+1}\not{\mathbb{D}}^{\mathbf{a}}\tilde{\varrho}_i)(t,\omega) - \sum_{i'=0}^{i-1}\frac{a_{i-i'-1}(i-1)i!}{i'!}\lim_{t\to\infty}(\mathbf{I}^{j-i+1}\not{\mathbb{D}}^{\mathbf{a}}\tilde{\varphi}_{i'})(t,\omega) = 0. \tag{8.32}$$

Thus, by induction, the $\tilde{\varphi}_i$ satisfy (8.24) for $m[\varrho] \leq i \leq l[\varrho]$. This then completes the proofs of (8.24) and (8.25).

Next, we consider the estimates of the flux and bulk integrals of $\tilde{\varphi}_i$. Since $\alpha_1[\varphi] < \alpha_1[\varrho] - 2 < 2l[\varrho] + 2$, the operators in $\mathbb{D}$ are linear combinations of operators in $\not{\mathbb{D}}$ and $R\partial_R$ with coefficients $O_\infty(1)$, and $R\partial_R$ commutes with the operators in $\not{\mathbb{D}}$, the above implies, for any $0 \leq i \leq l[\varrho]$, $q \in \{0,1\}$, and $t' \geq t_0$,

$$\begin{aligned}
\|r^{-l[\varrho]}\mathcal{L}_\xi^q\tilde{\varphi}_i\|^2_{W^{k-q}_{\alpha_1[\varphi]-3}(\Omega^{\text{ext}}_{t',\infty})} &\lesssim \sum_{j=0}^{k-q}\int_{t'}^\infty\int_t^\infty\int_{S^2}\left(r^{\alpha_1[\varphi]-3}|(R\partial_R)^j(r^{-l[\varrho]}\mathcal{L}_\xi^q\tilde{\varphi}_i)|^2_{k-q-j,\not{\mathbb{D}}}\right)\mathrm{d}^2\mu\mathrm{d}r\mathrm{d}t\\
&\lesssim \int_{\Omega^{\text{ext}}_{t',\infty}} r^{\alpha_1[\varphi]-3-2l[\varrho]}|\mathcal{L}_\xi^q\tilde{\varphi}_i|^2_{k-q,\mathbb{D}}\mathrm{d}^4\mu\\
&\lesssim D[\varrho]^2\int_{t'}^\infty\int_t^\infty r^{\alpha_1[\varphi]-3-2l[\varrho]}t^{2i-\alpha_1[\varrho]+3-2q}\mathrm{d}r\mathrm{d}t\\
&\lesssim D[\varrho]^2(t')^{-\delta-2q}.
\end{aligned}\tag{8.33}$$

By the same argument, it follows that

$$\begin{aligned}
\|r^{-l[\varrho]}\mathcal{L}_\xi^q\tilde{\varphi}_i\|^2_{W^{k-q}_{\alpha_1[\varphi]-2}(\Xi_{t',\infty})} &\lesssim \sum_{j=0}^{k-q}\int_{t'}^\infty\int_{S^2}\left(r^{\alpha_1[\varphi]-2}|(R\partial_R)^j(r^{-l[\varrho]}\mathcal{L}_\xi^q\tilde{\varphi}_i)|^2_{k-q-j,\not{\mathbb{D}}}\right)\Big|_{r=t}\mathrm{d}^2\mu\mathrm{d}t\\
&\lesssim \int_{t'}^\infty\int_{S^2} t^{\alpha_1[\varphi]-2-2l[\varrho]}|\mathcal{L}_\xi^q\tilde{\varphi}_i|^2_{k-q,\not{\mathbb{D}}}\mathrm{d}^2\mu\mathrm{d}t,
\end{aligned}\tag{8.34}$$

where in the last step we used the fact that $\tilde{\varphi}_i$ is independent of r. Hence, for any $0 \leq i \leq l[\varrho]$, $q \in \{0,1\}$, and $t' \geq t_0$

$$\begin{aligned}
\|r^{-l[\varrho]}\mathcal{L}_\xi^q\tilde{\varphi}_i\|^2_{W^{k-q}_{\alpha_1[\varphi]-2}(\Xi_{t',\infty})} &\lesssim \int_{t'}^\infty D[\varrho]^2t^{\alpha_1[\varphi]-2-2l[\varrho]}t^{2i-\alpha_1[\varrho]+3-2q}\mathrm{d}t\\
&\lesssim D[\varrho]^2(t')^{-\delta-2q}.
\end{aligned}\tag{8.35}$$

Gathering together these estimates for $\tilde{\varphi}_i$, we obtain for any $0 \leq i \leq l[\varrho]$, $q \in \{0,1\}$, and $t \geq t_0$,

$$\|r^{-l[\varrho]}\mathcal{L}_\xi^q\tilde{\varphi}_i\|^2_{W^{k-q}_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})} + \|r^{-l[\varrho]}\mathcal{L}_\xi^q\tilde{\varphi}_i\|^2_{W^{k-q}_{\alpha_1[\varphi]-3}(\Omega^{\text{ext}}_{t,\infty})} \lesssim D[\varrho]^2t^{-\delta-2q}. \tag{8.36}$$

Similarly, for any $0 \leq i \leq l[\varrho]$, $q \in \{0,1\}$, and $t \geq t_0$,

$$\|r^{-l[\varrho]-1}\mathcal{L}_\xi^q\tilde{\varphi}_i\|^2_{W^{k-q}_{\alpha_1[\varphi]+2-3}(\Omega^{\text{ext}}_{t,\infty})} \lesssim D[\varrho]^2t^{-\delta-2q}. \tag{8.37}$$

Step 2: Treat the φ_i. If $m[\varrho] \geq l[\varrho]$, it follows from (8.25) that $\tilde{\varphi}_i = 0$ for any $0 \leq i \leq m[\varrho]-1$ and hence formula (8.9a) implies $\varphi_i = 0$ for all $i \in \{0,\ldots,l[\varrho]-1\}$. Instead, if $m[\varrho] \leq l[\varrho]-1$, it follows from equations (8.25) and (8.9a) that $\varphi_i = 0$ for any $i \in \{0,\ldots,m[\varrho]-1\}$. Therefore, in either case, $\varphi_i = 0$ for any $i \in \{0,\ldots,m[\varphi]-1\}$ and any (t,ω). This proves condition (8.4).

For any $m[\varrho] \leq i \leq l[\varrho]-1 = l[\varphi]$, $t \in \mathbb{R}$, and $q \in \{0,1\}$, since $\alpha_1[\varrho] > 2l[\varrho]+3$, equations (8.9a), (8.24), and (8.25) can be used to obtain

$$\begin{aligned}
\int_{S^2}|\mathcal{L}_\xi^q\varphi_i(t,\omega)|^2_{k-q,\not{\mathbb{D}}}\mathrm{d}^2\mu &\lesssim \sum_{j=0}^{i}\int_{S^2}|\mathcal{L}_\xi^q\tilde{\varphi}_j(t,\omega)|^2_{k-q,\not{\mathbb{D}}}\mathrm{d}^2\mu \lesssim D[\varrho]^2\sum_{j=0}^{i}\langle t\rangle^{2j-\alpha_1[\varphi]+1-\delta-2q}\\
&\lesssim D[\varrho]^2\langle t\rangle^{2i-\alpha_1[\varrho]+3-2q},
\end{aligned}\tag{8.38a}$$

$$\lim_{t\to\infty}(\mathbf{I}^{j-i}\not{\mathbb{D}}^{\mathbf{a}}\varphi_i)(t,\omega) = \sum_{i'=0}^{i}\frac{i!a_{i-i'}}{(i')!}\lim_{t\to\infty}(\mathbf{I}^{j-i}\not{\mathbb{D}}^{\mathbf{a}}\tilde{\varphi}_{i'})(t,\omega) = 0, \qquad m[\varrho] \leq i < j \leq l[\varrho]. \tag{8.38b}$$

In particular, the estimate (8.38a) holds for any $0 \le i \le l[\varphi]$. These together verify the conditions (8.3d) and (8.3e).

The estimates for $\tilde{\varphi}_i$ in the above step, together with the uniform boundedness of the coefficients $\frac{i!a_{i-k}}{k!}$ in the expression (8.9a) of φ_i, imply that for any $0 \le i \le l[\varphi]$, $q \in \{0,1\}$ and $t \ge t_0$,

$$\|r^{-i}\mathcal{L}_\xi^q \varphi_i\|^2_{W^{k-q}_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})} + \|r^{-i}\mathcal{L}_\xi^q \varphi_i\|^2_{W^{k-q}_{\alpha_1[\varphi]-3}(\Omega^{\text{ext}}_{t,\infty})} \lesssim D[\varrho]^2 t^{-\delta-2q}. \tag{8.39}$$

Step 3: Treat $\tilde{\varphi}_{\text{rem};l[\varrho]}$. Since each $b'_{j-i-1}(R)R + jb_{j-i-1}(R)$ is uniformly bounded, from estimates (5.22c) and (5.22d) in lemma 5.4 about transport equations, one finds that, for any $q \in \{0,1\}$ and $t_2 \ge t_1 \ge t_0$,

$$\begin{aligned}
&\|\tilde{\varphi}_{\text{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2}(\Xi_{t_1,t_2})} + \|\tilde{\varphi}_{\text{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2}(\Sigma^{\text{ext}}_{t_2})} + \|\tilde{\varphi}_{\text{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-3}(\Omega^{\text{ext}}_{t_1,t_2})} \\
&\quad \lesssim \|\tilde{\varphi}_{\text{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2}(\Sigma^{\text{ext}}_{t_1})} + \|\varrho_{\text{rem};l[\varrho]}\|^2_{W^k_{(\alpha_1[\varphi]+2)-3}(\Omega^{\text{ext}}_{t_1,t_2})} + \sum_{i=0}^{l[\varrho]}\|r^{-l[\varrho]-1}\tilde{\varphi}_i\|^2_{W^k_{(\alpha_1[\varphi]+2)-3}(\Omega^{\text{ext}}_{t_1,t_2})},
\end{aligned} \tag{8.40}$$

$$\begin{aligned}
&\|\tilde{\varphi}_{\text{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2}(\Sigma^{\text{ext}}_{t_0})} + \|\tilde{\varphi}_{\text{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-3}(\Omega^{\text{early}}_{\text{init},t_0})} \\
&\quad \lesssim \|\tilde{\varphi}_{\text{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2}(\Sigma_{\text{init}})} + \|\varrho_{\text{rem};l[\varrho]}\|^2_{W^k_{(\alpha_1[\varphi]+2)-3}(\Omega^{\text{early}}_{\text{init},t_0})} + \sum_{i=0}^{l[\varrho]}\|r^{-l[\varrho]-1}\tilde{\varphi}_i\|^2_{W^k_{(\alpha_1[\varphi]+2)-3}(\Omega^{\text{early}}_{\text{init},t_0})}.
\end{aligned} \tag{8.41}$$

From the assumption that $\alpha_1[\varphi]+2 < \alpha_1[\varrho]$, there is the bound $\|\varrho_{\text{rem};l[\varrho]}\|^2_{W^k_{(\alpha_1[\varphi]+2)-3}(\Omega^{\text{ext}}_{t_1,t_2})} \lesssim D[\varrho]^2$ for the second term on the right of (8.40). The third term on the right of (8.40) are bounded by $D[\varrho]^2$ in estimate (8.37). Thus, one finds, for any $t \ge t_0$,

$$\begin{aligned}
&\|\tilde{\varphi}_{\text{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})} + \|\tilde{\varphi}_{\text{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2}(\Sigma^{\text{ext}}_{t})} + \|\tilde{\varphi}_{\text{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-3}(\Omega^{\text{ext}}_{t,\infty})} \\
&\quad \lesssim D[\varrho]^2 + \|\tilde{\varphi}_{\text{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2}(\Sigma^{\text{ext}}_{t_0})}.
\end{aligned} \tag{8.42}$$

From the assumption that ϱ has a $(k[\varrho], l[\varrho], m[\varrho], \alpha_1[\varrho], D[\varrho]^2)$ expansion and estimates (8.31) and (8.36) for $\tilde{\varphi}_i$, it follows that

$$\|\varrho_{\text{rem};l[\varrho]}\|^2_{W^k_{(\alpha_1[\varphi]+2)-3}(\Omega^{\text{early}}_{\text{init},t_0})} \lesssim D[\varrho]^2, \tag{8.43a}$$

$$\|\varrho_{\text{rem};l[\varrho]}\|^2_{W^k_{(\alpha_1[\varphi]+2)-3}(\Omega^{\text{ext}}_{t,\infty})} \lesssim D[\varrho]^2, \tag{8.43b}$$

$$\sum_{i=0}^{l[\varrho]}\|r^{-l[\varrho]-1}\tilde{\varphi}_i\|^2_{W^k_{(\alpha_1[\varphi]+2)-3}(\Omega^{\text{early}}_{\text{init},t_0})} \lesssim D[\varrho]^2, \tag{8.43c}$$

$$\sum_{i=0}^{l[\varrho]}\left(\|r^{-l[\varrho]-1}\tilde{\varphi}_i\|^2_{W^k_{(\alpha_1[\varphi]+2)-2}(\Xi_{t,\infty})} + \|r^{-l[\varrho]-1}\tilde{\varphi}_i\|^2_{W^k_{(\alpha_1[\varphi]+2)-3}(\Omega^{\text{ext}}_{t,\infty})}\right) \lesssim D[\varrho]^2. \tag{8.43d}$$

Moreover, it holds that $r \sim -t$ on Σ_{init}, and, since $\alpha_1[\varphi] < \alpha_1[\varrho] - 2 < 2l[\varrho]+2$, it follows that

$$\begin{aligned}
\sum_{i=0}^{l[\varrho]}\|r^{-i}\varrho_i\|^2_{W^{k-1}_{\alpha_1[\varphi]}(\Sigma_{\text{init}})} &\lesssim D[\varrho]^2 \int_{r_+}^{\infty} r^{\alpha_1[\varphi]} r^{-2i} r^{2i-\alpha_1[\varrho]+1}\mathrm{d}r \\
&\lesssim D[\varrho]^2,
\end{aligned} \tag{8.44a}$$

$$\begin{aligned}
\sum_{i=0}^{l[\varrho]}\|R^{l[\varrho]+1}\tilde{\varphi}_i(t,\omega)\|^2_{W^{k-1}_{\alpha_1[\varphi]}(\Sigma_{\text{init}})} &\lesssim D[\varrho]^2 \sum_{i=0}^{l[\varrho]}\int_{r_+}^{\infty} r^{\alpha_1[\varphi]} r^{-2l[\varrho]-2} r^{2i-\alpha_1[\varrho]+3}\mathrm{d}r \\
&\lesssim D[\varrho]^2.
\end{aligned} \tag{8.44b}$$

Since $\tilde{\varphi}_{\text{rem};l[\varrho]}$ vanishes on Σ_{init} by assumption, all the derivatives tangent to Σ_{init} of $\tilde{\varphi}_{\text{rem};l[\varrho]}$ also vanish. Each of the operators in $\mathbb{D}$ on Σ_{init} can be written as a sum of the tangential derivatives and $O_\infty(1)rY$. Therefore, we have from the expression (8.11) and estimates (8.44) that

$$\begin{aligned}
\|\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2}(\Sigma_{\mathrm{init}})} &\lesssim \|Y\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^{k-1}_{\alpha_1[\varphi]}(\Sigma_{\mathrm{init}})} \\
&\lesssim \|\varrho_{\mathrm{rem};l[\varrho]}\|^2_{W^{k-1}_{\alpha_1[\varphi]}(\Sigma_{\mathrm{init}})} + \sum_{i=0}^{l[\varrho]} \|R^{l[\varrho]+1}\tilde{\varphi}_i(t,\omega)\|^2_{W^{k-1}_{\alpha_1[\varphi]}(\Sigma_{\mathrm{init}})} \\
&\lesssim D[\varrho]^2. \qquad (8.45)
\end{aligned}$$

Combining estimates (8.40), (8.41), (8.42), (8.43), and (8.45) gives that, for any $t \geq t_0$,

$$\|\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})} + \|\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2}(\Sigma^{\mathrm{ext}}_t)} + \|\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-3}(\Omega^{\mathrm{ext}}_{t,\infty})} \lesssim D[\varrho]^2, \qquad (8.46a)$$

$$\|\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2}(\Sigma^{\mathrm{ext}}_{t_0})} + \|\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-3}(\Omega^{\mathrm{early}}_{\mathrm{init},t_0})} \lesssim D[\varrho]^2. \qquad (8.46b)$$

An application of lemma 4.34 together with estimate (8.43b) implies

$$\|\varrho_{\mathrm{rem};l[\varrho]}\|^2_{W^{k-1}_{(\alpha_1[\varphi]+2)-2}(\Xi_{t,\infty})} \lesssim \|\varrho_{\mathrm{rem};l[\varrho]}\|^2_{W^k_{(\alpha_1[\varphi]+2)-3}(\Omega^{\mathrm{ext}}_{t,\infty})} \lesssim D[\varrho]^2. \qquad (8.46c)$$

Hence, from the assumption, equation (8.10), and estimates (8.43), we have, for any $t \geq t_0$,

$$\begin{aligned}
&\|Y\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^{k-1}_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})} + \|Y\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-3}(\Omega^{\mathrm{ext}}_{t,\infty})} \\
&\lesssim t^{-2}\left(\|Y\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^{k-1}_{(\alpha_1[\varphi]+2)-2}(\Xi_{t,\infty})} + \|Y\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^k_{(\alpha_1[\varphi]+2)-3}(\Omega^{\mathrm{ext}}_{t,\infty})}\right) \\
&\lesssim D[\varrho]^2 t^{-2}, \qquad (8.46d)
\end{aligned}$$

which follows from (8.43d), (8.46c), and the fact that $r \geq t$ in the exterior region. It then holds that, for any $t \geq t_0$,

$$\begin{aligned}
&\|\mathcal{L}_\xi\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^{k-1}_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})} + \|\mathcal{L}_\xi\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^{k-1}_{\alpha_1[\varphi]-3}(\Omega^{\mathrm{ext}}_{t,\infty})} \\
&\quad\lesssim \|Y\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^{k-1}_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})} + \|Y\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^{k-1}_{\alpha_1[\varphi]-3}(\Omega^{\mathrm{ext}}_{t,\infty})} \\
&\qquad + \|\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2-2}(\Xi_{t,\infty})} + \|\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2-3}(\Omega^{\mathrm{ext}}_{t,\infty})} \\
&\quad\lesssim D[\varrho]^2 t^{-2}. \qquad (8.47)
\end{aligned}$$

Hence, together with (8.46a), this implies, for any $q \in \{0,1\}$ and $t \geq t_0$,

$$\|\mathcal{L}^q_\xi\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^{k-q}_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})} + \|\mathcal{L}^q_\xi\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^{k-q}_{\alpha_1[\varphi]-3}(\Omega^{\mathrm{ext}}_{t,\infty})} \lesssim D[\varrho]^2 t^{-2q}. \qquad (8.48)$$

For any $t \geq t_0+1$, there exists an $i \in \mathbb{N}$ such that $t \in [t_0+2^i, t_0+2^{i+1}]$. We apply the mean-value principle to the first term of (8.48), with the time interval replaced by $[t_0+2^i, t_0+2^{i+1}]$, to conclude there exists a $t_{(i)} \in [t_0+2^i, t_0+2^{i+1}]$ such that

$$\int_{S^2} |t_{(i)}^{\frac{\alpha_1[\varphi]-2}{2}}\tilde{\varphi}_{\mathrm{rem};l[\varrho]}(t_{(i)}, t_{(i)}, \omega)|^2_{k,\mathbb{D}} \mathrm{d}^2\mu \lesssim D[\varrho]^2(t_0+2^i)^{-1} \lesssim D[\varrho]^2 t^{-1}. \qquad (8.49)$$

From fundamental theorem of calculus,

$$\begin{aligned}
\int_{S^2} |t^{\frac{\alpha_1[\varphi]-2}{2}}\tilde{\varphi}_{\mathrm{rem};l[\varrho]}(t,t,\omega)|^2_{k-1,\mathbb{D}} \mathrm{d}^2\mu &\lesssim \int_{S^2} |t_{(i)}^{\frac{\alpha_1[\varphi]-2}{2}}\tilde{\varphi}_{\mathrm{rem};l[\varrho]}(t_{(i)}, t_{(i)}, \omega)|^2_{k-1,\mathbb{D}} \mathrm{d}^2\mu \\
&\quad + \|\mathcal{L}_\xi\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^{k-1}_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})} + \|\tilde{\varphi}_{\mathrm{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2-2}(\Xi_{t,\infty})} \\
&\lesssim D[\varrho]^2 t^{-1}. \qquad (8.50)
\end{aligned}$$

Similarly we have for $t \in [t_0, t_0+1]$ that $\int_{S^2} |\tilde{\varphi}_{\mathrm{rem};l[\varrho]}(t,t,\omega)|^2_{k-1,\mathbb{D}} \mathrm{d}^2\mu \lesssim D[\varrho]^2$. Therefore, for any $t \geq t_0$,

$$\int_{S^2} |t^{\frac{\alpha_1[\varphi]-1}{2}}\tilde{\varphi}_{\mathrm{rem};l[\varrho]}(t,t,\omega)|^2_{k-1,\mathbb{D}} \mathrm{d}^2\mu \lesssim D[\varrho]^2. \qquad (8.51)$$

Notice from (4.50) and (8.46a), we have in the exterior region that, for any $t \geq t_0$,

$$\begin{aligned}
&\int_{S^2} |r^{\frac{\alpha_1[\varphi]-1}{2}} \tilde{\varphi}_{\text{rem};l[\varrho]}(t,r,\omega)|^2_{k-1,\mathbb{D}} \mathrm{d}^2\mu \\
&\lesssim \int_{S^2} |r^{\frac{\alpha_1[\varphi]-1}{2}} \tilde{\varphi}_{\text{rem};l[\varrho]}(t,t,\omega)|^2_{k-1,\mathbb{D}} \mathrm{d}^2\mu + \|\tilde{\varphi}_{\text{rem};l[\varrho]}\|^2_{W^k_{\alpha_1[\varphi]-2}(\Sigma_t^{\text{ext}})} \\
&\lesssim D[\varrho]^2. \qquad (8.52)
\end{aligned}$$

From lemma 4.27, the following pointwise estimates then hold for any $t \geq t_0$ in the exterior region

$$|\tilde{\varphi}_{\text{rem};l[\varrho]}|^2_{k-3,\mathbb{D}} \lesssim D[\varrho]^2 r^{-(\alpha_1[\varphi]-1)}. \qquad (8.53)$$

Step 4: Treat $\tilde{\varphi}_{\text{hom};l[\varrho]}$. Given a point $p \in \Omega^{\text{ext}}_{t_1,t_2}$ with coordinates (t,r,ω), let γ denote the integral curve along Y through the point. The value of $\mathcal{L}^q_\xi \tilde{\varphi}_{\text{hom};l[\varrho]}$ is constant along γ, so its value at p is equal to its value at the intersection of γ and Σ_{init}. Since the rates of change of $-t$ and r are comparable along γ, it follows that the coordinates $(\tilde{t},\tilde{r},\tilde{\omega})$ of the intersection of γ and Σ_{init} satisfy $-\tilde{t} \sim \tilde{r} \sim t+2r$. From the decay rates for $\mathcal{L}^q_\xi \varphi_{\text{init}}$ and $\mathcal{L}^q_\xi \tilde{\varphi}_i$, and since $\alpha_1[\varrho]-3 < 2l[\varrho]+1 = 2l[\varphi]+3$ and $\alpha_1[\varphi] = \alpha_1[\varrho]-2-\delta$, one finds for any $q \in \{0,1\}$,

$$\begin{aligned}
\int_{S^2} |\mathcal{L}^q_\xi \tilde{\varphi}_{\text{hom};l[\varrho]}|^2 \mathrm{d}^2\mu &\lesssim (t+2r)^{-\alpha_1[\varrho]+3-2q} \mathbb{P}^{k[\varphi];\alpha_1[\varrho]-3}_{\text{init}}(\varphi) + D[\varrho]^2 (t+2r)^{-\alpha_1[\varrho]+3-2q} \\
&\lesssim (t+2r)^{-\alpha_1[\varphi]+1-\delta-2q} \left(\mathbb{P}^{k[\varphi];2l[\varphi]+3}_{\text{init}}(\varphi) + D[\varrho]^2 \right). \qquad (8.54)
\end{aligned}$$

The quantity $\mathbb{P}^{k[\varphi];2l[\varphi]+3}_{\text{init}}(\varphi)$ in the above estimates can be replaced by $\mathbb{I}^{k[\varphi]+1;2l[\varphi]+3}_{\text{init}}(\varphi)$ from lemma 4.36, implying that

$$\int_{S^2} |\mathcal{L}^q_\xi \tilde{\varphi}_{\text{hom};l[\varrho]}|^2 \mathrm{d}^2\mu \lesssim (t+2r)^{-\alpha_1[\varphi]+1-\delta-2q} D[\varphi]^2. \qquad (8.55)$$

Applying a Y derivative to $\mathcal{L}^q_\xi \tilde{\varphi}_{\text{hom};l[\varrho]}$ gives zero. Differentiating along rV or applying $\mathring{\eth}$ or $\mathring{\eth}'$ corresponds to differentiating along a vector of length r on the initial data. Since derivatives decay one power faster, this means that $\int_{S^2} |\mathcal{L}^q_\xi \tilde{\varphi}_{\text{hom};l[\varrho]}|^2_{k-q,\mathbb{D}} \mathrm{d}^2\mu$ decays at the same rate, although the constant depends on the k norm, i.e. for any $q \in \{0,1\}$,

$$\int_{S^2} |\mathcal{L}^q_\xi \tilde{\varphi}_{\text{hom};l[\varrho]}|^2_{k-q,\mathbb{D}} \mathrm{d}^2\mu \lesssim (t+2r)^{-\alpha_1[\varphi]+1-\delta-2q} D[\varphi]^2. \qquad (8.56)$$

As with the $\mathcal{L}^q_\xi \tilde{\varphi}_i$, since $\alpha_1[\varphi] < \alpha_1[\varrho]-2$, one finds that, for any $q \in \{0,1\}$ and $t' \geq t_0$,

$$\begin{aligned}
\|\mathcal{L}^q_\xi \tilde{\varphi}_{\text{hom};l[\varrho]}\|^2_{W^{k-q}_{\alpha_1[\varphi]-3}(\Omega^{\text{ext}}_{t,\infty})} &\lesssim \int_t^\infty \int_{t'}^\infty D[\varphi]^2 r^{\alpha_1[\varphi]-3} (t'+2r)^{-\alpha_1[\varphi]+1-\delta-2q} \mathrm{d}r \mathrm{d}t' \\
&\lesssim D[\varphi]^2 t^{-\delta-2q}, \qquad (8.57\text{a}) \\
\|\mathcal{L}^q_\xi \tilde{\varphi}_{\text{hom};l[\varrho]}\|^2_{W^{k-q}_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})} &\lesssim \int_t^\infty D[\varphi]^2 r^{\alpha_1[\varphi]-2} (3r)^{-\alpha_1[\varphi]+1-\delta-2q} \mathrm{d}t' \\
&\lesssim D[\varphi]^2 t^{-\delta-2q}, \qquad (8.57\text{b}) \\
\|\mathcal{L}^q_\xi \tilde{\varphi}_{\text{hom};l[\varrho]}\|^2_{W^{k-q}_{\alpha_1[\varphi]-2}(\Sigma^{\text{ext}}_t)} &\lesssim \int_t^\infty D[\varphi]^2 r^{\alpha_1[\varphi]-2} (t+2r)^{-\alpha_1[\varphi]+1-\delta-2q} \mathrm{d}r \\
&\lesssim D[\varphi]^2 t^{-\delta-2q}, \qquad (8.57\text{c}) \\
\|\mathcal{L}^q_\xi \tilde{\varphi}_{\text{hom};l[\varrho]}\|^2_{W^{k-q}_{\alpha_1[\varphi]-3}(\Omega^{\text{early}}_{\text{init},t_0})} &\lesssim \int_{-\infty}^{t_0} \int_{|t'|}^\infty D[\varphi]^2 r^{\alpha_1[\varphi]-3} (t'+2r)^{-\alpha_1[\varphi]+1-\delta-2q} \mathrm{d}r \mathrm{d}t' \\
&\lesssim D[\varphi]^2. \qquad (8.57\text{d})
\end{aligned}$$

Step 5: Treat $\varphi_{\text{rem};l[\varphi]}$. One can combine the results for the $\{\mathcal{L}^q_\xi \tilde{\varphi}_i\}_{i=0}^{l[\varrho]}$, for $\mathcal{L}^q_\xi \tilde{\varphi}_{\text{rem};l[\varrho]}$, and for $\mathcal{L}^q_\xi \tilde{\varphi}_{\text{hom};l[\varrho]}$. Combining these bounds with uniform bounds on $b_{j-i-1}(R)$, and noticing $l[\varphi] =$

$l[\varrho]-1$, one finds, for any $q\in\{0,1\}$ and $t\geq t_0$,

$$\|\mathcal{L}_\xi^q\varphi_{\mathrm{rem};l[\varphi]}\|^2_{W^{k-q}_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})}+\|\mathcal{L}_\xi^q\varphi_{\mathrm{rem};l[\varphi]}\|^2_{W^{k-q}_{\alpha_1[\varphi]-3}(\Omega^{\mathrm{ext}}_{t,\infty})}\lesssim D[\varphi]^2t^{-2q}, \tag{8.58a}$$

$$\|\varphi_{\mathrm{rem};l[\varphi]}\|^2_{W^{k}_{\alpha_1[\varphi]-3}(\Omega^{\mathrm{early}}_{\mathrm{init},t_0})}\lesssim D[\varphi]^2. \tag{8.58b}$$

From lemma 4.27 and rewriting rV using equation (4.8), the estimates of the $L^2(S^2)$ norm of $\{\mathcal{L}_\xi^q\tilde{\varphi}_i\}_{i=0}^{l[\varrho]}$ and $\mathcal{L}_\xi^q\tilde{\varphi}_{\mathrm{hom};l[\varrho]}$ in inequalities (8.24a) and (8.56) imply that in the exterior region, for any $t\geq t_0$, $i\in\{m[\varrho]+1,\dots,l[\varrho]\}$, and $n\in\mathbb{N}$,

$$\begin{aligned}
|R^n\tilde{\varphi}_i|^2_{k-2,\mathbb{D}}&\lesssim\sum_{j=0}^{k-2}|(rV)^j(R^n\tilde{\varphi}_i)|^2_{k-2-j,\not{\mathbb{D}}}\\
&\lesssim\sum_{j=0}^{k-2}|(R\partial_R)^j(R^n)\tilde{\varphi}_i|^2_{k-2-j,\not{\mathbb{D}}}\\
&\lesssim\int_{S^2}R^{2n}|\tilde{\varphi}_i(t,\omega)|^2_{k,\not{\mathbb{D}}}\mathrm{d}^2\mu\\
&\lesssim D[\varrho]^2R^{2n}t^{2i-\alpha_1[\varrho]+3}, &(8.59)\\
|\tilde{\varphi}_{\mathrm{hom};l[\varrho]}|^2_{k-2,\mathbb{D}}&\lesssim D[\varphi]^2R^{\alpha_1[\varrho]-3}. &(8.60)
\end{aligned}$$

Together with the pointwise estimates of $\tilde{\varphi}_{\mathrm{rem};l[\varrho]}$ and the uniform boundedness of $b_{l[\varrho]-i-1}(R)$, it follows that in the exterior region, for any $t\geq t_0$,

$$\text{for } m[\varrho]\leq l[\varrho],\qquad |\varphi_{\mathrm{rem};l[\varphi]}|^2_{k-3,\mathbb{D}}\lesssim D[\varphi]^2r^{-2l[\varrho]}t^{-\alpha_1[\varphi]+1+2l[\varrho]}, \tag{8.61a}$$

$$\text{for } m[\varrho]=l[\varrho]+1,\qquad |\varphi_{\mathrm{rem};l[\varphi]}|^2_{k-3,\mathbb{D}}\lesssim D[\varphi]^2r^{-\alpha_1[\varphi]+1}. \tag{8.61b}$$

Step 6: Treat φ. Combining inequalities (8.39) and (8.58a) for the $\{\mathcal{L}_\xi^q\varphi_i\}_{i=0}^{l[\varphi]}$ and $\mathcal{L}_\xi^q\varphi_{\mathrm{rem};l[\varphi]}$ gives, for any $q\in\{0,1\}$ and $t\geq t_0$,

$$\begin{aligned}
\|\mathcal{L}_\xi^q\varphi\|^2_{W^{k-q}_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})}&\lesssim\sum_{i=0}^{l[\varphi]}\|r^{-i}\mathcal{L}_\xi^q\varphi_i\|^2_{W^{k-q}_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})}+\|\varphi_{\mathrm{rem};l[\varphi]}\|^2_{W^{k-q}_{\alpha_1[\varphi]-2}(\Xi_{t,\infty})}\\
&\lesssim D[\varphi]^2t^{-2q}. &(8.62)
\end{aligned}$$

For any $i\leq m[\varrho]-1$, $\varphi_i=0$, and for any $m[\varrho]\leq i\leq l[\varrho]-1=l[\varphi]$, we have from (8.59) with $n=i$ and the uniform boundedness of $\{a_i\}_{i=0}^{l[\varphi]}$ that in the exterior region, for any $t\geq t_0$,

$$|R^i\varphi_i|^2_{k-2,\mathbb{D}}\lesssim\sum_{j=0}^{i}|R^i\tilde{\varphi}_j|^2_{k-2,\mathbb{D}}\lesssim\sum_{j=m[\varphi]+1}^{i}r^{-2i}t^{2j-\alpha_1[\varrho]+3}\lesssim D[\varrho]^2R^{2i}t^{2i-\alpha_1[\varrho]+3}. \tag{8.63}$$

We have then from the pointwise estimate for $\varphi_{\mathrm{rem};l[\varphi]}$, the fact that the $\{\varphi_i\}_{i=0}^{m[\varrho]-1}$ vanish and the above pointwise estimates for $\{\varphi_i\}_{i=m[\varrho]}^{l[\varphi]}$ that in the exterior region, for any $t\geq t_0$,

$$\text{for } m[\varrho]\leq l[\varrho],\qquad |\varphi|^2_{k-3,\mathbb{D}}\lesssim D[\varphi]^2r^{-2m[\varrho]}t^{-m[\varphi]+1+2\alpha_1[\varrho]}, \tag{8.64a}$$

$$\text{for } m[\varrho]=l[\varrho]+1,\qquad |\varphi|^2_{k-3,\mathbb{D}}\lesssim D[\varphi]^2r^{-\alpha_1[\varphi]+1}. \tag{8.64b}$$

Therefore, we conclude that φ has a $(k[\varrho],l[\varrho],m[\varphi],\alpha_1[\varphi],D[\varphi]^2)$ expansion, and, for any $q\in\{0,1\}$ and $t\geq t_0$, the estimates (8.22) and (8.23) hold true. □

Lemma 8.7 (Transformations of expansions)**.** *Let $k,l,m\in\mathbb{N}$ be such that $0\leq m\leq l+1$. Let α_1 be such that $2l+3<\alpha_1<2l+4$. Let $D>0$. Let ϱ be a spin-weighted scalar.*

(1) If $0\leq k'\leq k$, $0\leq m'\leq m$ and ϱ has a (k,l,m,α_1,D^2) expansion, then ϱ has a (k',l,m',α_1,D^2) expansion.
(2) If ϱ_1 and ϱ_2 both have (k,l,m,α_1,D^2) expansions, then $\varrho_1+\varrho_2$ has a (k,l,m,α_1,D^2) expansion.

(3) *Let $n \in \mathbb{Z}$, $n+m \geq 0$, and $n+l \geq 0$. Let f be a homogeneous rational function of r, $\sqrt{r^2+a^2}$, κ_1, and $\bar{\kappa}_{1'}$ of degree $-n$ that has no singularities for $R \in [0, R_0^{-1}]$. Then there is a constant $C_f > 0$ such that if ϱ has a (k, l, m, α_1, D^2) expansion, then $f\varrho$ has a $(k, l+n, m+n, \alpha_1+2n, C_f D^2)$ expansion.*

(4) *If ϱ has a (k, l, m, α_1, D^2) expansion and has spin-weight s, then $\tau\varrho$ and $\bar{\tau}'\varrho$ have $(k, l+2, m+2, \alpha_1+4, D^2)$ expansions and have spin-weight $s+1$, and $\bar{\tau}\varrho$ and $\tau'\varrho$ have $(k, l+2, m+2, \alpha_1+4, D^2)$ expansions and have spin-weight $s-1$.*

(5) *If ϱ has a (k, l, m, α_1, D^2) expansion and has spin-weight s, then $\kappa_1 ð\, \varrho$ has a $(k-1, l, m, \alpha_1, D^2)$ expansion and has spin-weight $s+1$, and $\bar{\kappa}_{1'} ð'\, \varrho$ has a $(k-1, l, m, \alpha_1, D^2)$ expansion and has spin-weight $s-1$.*

Proof. If $k' \leq k$ and $m' \leq m$, then the condition to have a (k, l, m, α_1, D^2) expansion is strictly stronger than the condition to have a $(k', l, m', \alpha_1, D^2)$ expansion, so the former implies the latter, which implies point 1.

Point 2 follows directly from summing the expansions, summing the bounds, and noting the linearity in both the integrability condition (8.3e) and the vanishing condition (8.4).

Now consider point 3. Observe that if ϱ has a (k, l, m, α_1, D^2) expansion, then $\vartheta = r^{-n}\varrho$ has a $(k, l+n, m+n, \alpha_1+2n, D^2)$ expansion, where $\vartheta_i = 0$ for $i \leq n+m$, $\vartheta_i = \varrho_{i-m}$ for $i > n+m$, and $\vartheta_{\text{rem};l+n} = r^{-n}\varrho_{\text{rem};l}$. Thus, it is sufficient to show that if f is a homogeneous rational function of degree 0 and ϱ has a (k, l, m, α_1, D^2) expansion, then $f\varrho$ has a (k, l, m, α_1, D^2) expansion. Expanding f as an order l power series in R and multiplying the expansions for f and ϱ together, one obtains an order l expansion for $f\varrho$. Because f is rational with no singularities on $R = 0$, each of the expansion terms in f are smooth functions of the spherical coordinates alone. Thus, the expansion terms for $f\varrho$ have the same decay and t-integrability conditions as ϱ. The remainder term for f decays as r^{-l-1}. The remainder term for $f\varrho$ consists of products of expansion terms of f and of ϱ, of expansion terms of f and the remainder for ϱ, of the remainder for f and the expansion terms of ϱ, and of the remainder term for f and the remainder term for ϱ. The expansion terms for f and the remainder are all homogeneous rational functions without singularities in the region under consideration and with a characteristic rate of decay. Since f is t independent, $\mathcal{L}_\xi(f\varrho) = f\mathcal{L}_\xi\varrho$ and similarly for higher derivatives. All four types of products in the expansion of $f\varrho$ will have bounded integrals for $t \leq t_0$ when integrated over $\Omega^{\text{ext}}_{t,\infty}$. Thus, all the conditions for a (k, l, m, α_1, D^2) expansion are satisfied.

In point 4, the claim about the spin weight follows from properties of products of spin-weighted quantities. The bounds can be calculated in the Znajek tetrad using the argument from the previous paragraph and the fact that, in the Znajek tetrad, τ and τ' is $a\sin\theta$ times a homogeneous rational function in κ_1 and $\bar{\kappa}_{1'}$ of degree -2.

Similarly, in point 5, the claim about spin follows from the fact that κ_1 and $\bar{\kappa}_{1'}$ are spin-weight zero quantities, and $ð$ and $ð'$ are spin $+1$ and -1 operators. The bounds follow from the relations (2.32d) and (2.32d) that $\kappa_1 ð\,\varrho$ is a linear combination of $\mathring{ð}\,\varrho$, $\kappa_1^2\tau\mathcal{L}_\xi\varrho$, and $\kappa_1\tau\varrho$ and $\bar{\kappa}_{1'} ð'\,\varrho$ is a linear combination of $\mathring{ð}'\,\varrho$, $\bar{\kappa}_{1'}\bar{\tau}\mathcal{L}_\xi\varrho$, and $\bar{\kappa}_{1'}\bar{\tau}\varrho$, and the fact that the operators $\mathring{ð}$, $\mathring{ð}'$, and $\mathcal{L}_\xi$ are in $\mathbb{D}$, the number of which is measured by k. □

8.2. Integration on $\mathscr{I}^+$ and the Teukolsky-Starobinsky identity.

Definition 8.8 (Taylor expansion at $\mathscr{I}^+$)**.** Let $\hat{\psi}_{-2}, \hat{\psi}_{+2}$ be as in definition 8.1. Working in the compactified hyperboloidal coordinate system (t, R, θ, ϕ) and restricting to the Znajek tetrad, let the spin-weighted scalars A_i, $i = 0, \ldots, 3$, B_0 on $\mathscr{I}^+$ be the Taylor coefficients of $\hat{\psi}_{-2}, \hat{\psi}_{+2}$ defined by

$$A_i = \partial_R^i \hat{\psi}_{-2}\big|_{\mathscr{I}^+}, \quad i = 0, \ldots, 3, \tag{8.65a}$$

$$B_0 = \hat{\psi}_{+2}\big|_{\mathscr{I}^+}, \tag{8.65b}$$

and let $A_{\text{rem};3}, B_{\text{rem};0}$ be the corresponding remainder terms such that

$$\hat{\psi}_{-2} = \sum_{i=0}^{3} \frac{R^i}{i!} A_i(t,\omega) + A_{\text{rem};3}, \tag{8.66a}$$

$$\hat{\psi}_{+2} = B_0 + B_{\text{rem};0}. \tag{8.66b}$$

Lemma 8.9. *Let* $\mathbf{a}$ *be a multiindex. With* A_i*,* $i = 0, \dots, 3$*,* B_0 *as in definition 8.8, assume that*

$$\lim_{t\to-\infty} \mathcal{L}_\xi^j \not{D}^{\mathbf{a}} B_0 = \lim_{t\to\infty} \mathcal{L}_\xi^j \not{D}^{\mathbf{a}} B_0 = 0, \quad j = 0, \dots, 4 \tag{8.67a}$$

and

$$\lim_{t\to-\infty} \mathcal{L}_\xi^j \not{D}^{\mathbf{a}} A_0 = \lim_{t\to\infty} \mathcal{L}_\xi^j \not{D}^{\mathbf{a}} A_0 = 0, \quad j = 0, \dots, 4. \tag{8.67b}$$

Then with $\mathbf{I}$ *defined as in definition 8.3,*

$$\lim_{t\to\infty} \mathbf{I}^j \not{D}^{\mathbf{a}} A_0 = 0, \quad j = 1, \dots, 4. \tag{8.68}$$

Proof. We first prove the statement for $\mathbf{a} = 0$. Passing to the Znajek tetrad, we may replace $\mathcal{L}_\xi$ by ∂_t for spin-weighted scalars on $\mathscr{I}^+$. Equation (3.29) yields, after using the expression (2.75a) for Y and taking the limit $R \to 0$, that on $\mathscr{I}^+$,

$$\mathring{\eth}^4 A_0 = -3M\partial_t(\bar{A}_0) - \sum_{k=1}^{4} \binom{4}{k} \mathring{\tau}^k \mathring{\eth}^{4-k} \partial_t{}^k A_0 + 4\partial_t^4 B_0. \tag{8.69}$$

Integrating (8.69) j times from $t = -\infty$, we have by (8.67)

$$\mathring{\eth}^4 \mathbf{I}^j A_0 = -3M\mathbf{I}^j\partial_t(\bar{A}_0) - \sum_{k=1}^{4} \binom{4}{k} \mathring{\tau}^k \mathring{\eth}^{4-k} \mathbf{I}^j \partial_t^k A_0 + 4\mathbf{I}^j \partial_t^4 B_0, \quad j = 1, \dots, 4. \tag{8.70}$$

From definition 8.3 we have that for a function f satisfying (8.67),

$$\partial_t \mathbf{I} f = \mathbf{I}\partial_t f = f. \tag{8.71}$$

For $j = 1$ we have

$$\mathring{\eth}^4 \mathbf{I} A_0 = -3M\bar{A}_0 - \sum_{k=1}^{4} \binom{4}{k} \mathring{\tau}^k \mathring{\eth}^{4-k} \partial_t^{k-1} A_0 + 4\partial_t^3 B_0. \tag{8.72}$$

Recall that A, and hence A_0 has spin-weight -2. Acting on a spin-weighted spherical harmonic ${}_{-2}Y_{lm}$, we have

$$\mathring{\eth}^4 {}_{-2}Y_{lm} = \frac{(l+2)!}{4(l-2)!} {}_{+2}Y_{lm}, \tag{8.73}$$

and hence, since we may restrict to considering $l \geq 2$, we find that the operator $\mathring{\eth}^4$ has trivial kernel when acting on fields of spin-weight -2. Taking the limit $t \to \infty$ on both sides of (8.72), and after using (8.67) and the fact that $\mathring{\eth}^4$ has trivial kernel on spin-weighted functions on S^2 with spin-weight -2, this gives the statement for $j = 1$. For $j = 2, \dots, 4$, the statement can be proven in a similar manner, using induction with $j = 1$ as base. This proves the lemma for $\mathbf{a} = 0$.

We prove the lemma for $\mathbf{a} \neq 0$ by induction on $|\mathbf{a}|$, with $\mathbf{a} = 0$ as base. Thus, let $k \geq 1$ be an integer, and assume the lemma is proved for $|\mathbf{a}| = k - 1$. Applying $\not{D}^{\mathbf{a}}$ to both sides of (8.69) yields

$$\begin{aligned}\mathring{\eth}^4 \not{D}^{\mathbf{a}} A_0 = {}& [\mathring{\eth}^4, \not{D}^{\mathbf{a}}] A_0 - 3M\partial_t(\not{D}^{\mathbf{a}} \bar{A}_0) - \sum_{k=1}^{4} \binom{4}{k} \mathring{\tau}^k \mathring{\eth}^{4-k} \partial_t{}^k \not{D}^{\mathbf{a}} A_0 \\ & - \sum_{k=1}^{4} \binom{4}{k} \partial_t^k [\not{D}^{\mathbf{a}}, \mathring{\tau}^k \mathring{\eth}^{4-k}] A_0 + 4\partial_t^4 \not{D}^{\mathbf{a}} B_0. \end{aligned} \tag{8.74}$$

The commutators on the right-hand side of (8.74) can be evaluated by noting that ∂_t commutes with $\mathring{\eth}$ and making use of the identities (3.28) and the commutation formula (2.42d). By the induction hypothesis, we have that each term on the right-hand side satisfies the assumptions of the lemma. Therefore, we can proceed as above and inductively prove (8.68) for $j = 1, \dots, 4$. This completes the proof of the lemma. □

Lemma 8.10. *Let* $\mathbf{a}$ *be a multiindex, and let the assumptions in lemma 8.9 hold. Assume that*

$$\lim_{t\to-\infty} \not{D}^{\mathbf{a}} A_i(t,\omega) = 0, \quad i = 0,\dots,3. \tag{8.75}$$

Then

$$\lim_{t\to\infty} \mathbf{I}^j \not{D}^{\mathbf{a}} A_i(t,\omega) = 0, \quad i = 0,\dots,3, \quad j = 0,\dots,4-i. \tag{8.76}$$

Proof. We first consider the case $\mathbf{a} = 0$. Taylor expanding the Teukolsky equation (3.25a) at $\mathscr{I}^+$ and using (8.66) gives a recursive set of equations for $\mathcal{L}_\xi A_i$, $i = 0,\dots,3$, which after passing to the Znajek tetrad takes the form

$$\partial_t A_1 = 2A_0 + 4M\partial_t A_0 + 2C_{\text{hyp}}M^2\partial_t\partial_t A_0 + 2a\partial_t\partial_\phi A_0 - \tfrac{1}{2}S_{-2}(A_0), \tag{8.77a}$$

$$\begin{aligned}\partial_t A_2 = {} & -MA_0 + 3A_1 + 2(4 - C_{\text{hyp}})M^2\partial_t A_0 + 4M\partial_t A_1 + 2C_{\text{hyp}}M^2\partial_t\partial_t A_1 + 4Ma\partial_t\partial_\phi A_0 \\ & + 2a\partial_t\partial_\phi A_1 + a\partial_\phi A_0 - \tfrac{1}{2}S_{-2}(A_1) + (16M^3 - 4Ma^2 - \tfrac{1}{6}H^{(3)}(0))\partial_t\partial_t A_0,\end{aligned} \tag{8.77b}$$

$$\begin{aligned}\partial_t A_3 = {} & -3a^2A_0 + 3A_2 + (16M^2 - 2a^2)\partial_t A_1 + 4M\partial_t A_2 + 2C_{\text{hyp}}M^2\partial_t\partial_t A_2 \\ & + 4M^2a(4 - C_{\text{hyp}})\partial_t\partial_\phi A_0 + 8Ma\partial_t\partial_\phi A_1 + 2a\partial_t\partial_\phi A_2 + 4a\partial_\phi A_1 - \tfrac{1}{2}S_{-2}(A_2) \\ & + (32M^3 - 8Ma^2 - \tfrac{1}{3}H^{(3)}(0))\partial_t\partial_t A_1 + (16M^3 - 4C_{\text{hyp}}M^3 - 12Ma^2 + \tfrac{1}{6}H^{(3)}(0))\partial_t A_0 \\ & + (64M^4 - 4C_{\text{hyp}}^2M^4 - 32M^2a^2 + 4C_{\text{hyp}}M^2a^2 - \tfrac{1}{12}H^{(4)}(0))\partial_t\partial_t A_0.\end{aligned} \tag{8.77c}$$

The system (8.77) is of the form

$$\partial_t A_i = \mathbf{L}_i{}^k A_k \tag{8.78}$$

where, by inspection, $\mathbf{L}_i{}^j$ is a strictly lower-triangular matrix of operators on $\mathscr{I}^+$ with entries which are linear combinations of symmetry operators of order up to two of the TME, i.e. S_{-2}, $\partial_t^2, \partial_t\partial_\phi, \partial_t, \partial_\phi$ and constants. The coefficients are bounded constants and depend only on M, a, C_{hyp}, and the Taylor terms $H^3(0)$ and $H^4(0)$, where H is given by (2.73) and (2.48).

From (8.78) we get the recursion relation

$$A_i(t,\omega) = \lim_{t\to-\infty} A_i(t,\omega) + \int_{-\infty}^t \sum_{k=0}^{i-1} \mathbf{L}_i{}^k A_k(t',\omega)\mathrm{d}t', \quad i = 1,2,3. \tag{8.79}$$

Lemma 8.9 shows that the $i = 0$ case of (8.76) is valid. We consider the case $i = 1$. From (8.79) and (8.75) we have

$$\lim_{t\to\infty} \mathbf{I}^j A_1 = \int_{-\infty}^{\infty} \mathbf{L}_1{}^0 \mathbf{I}^j A_0(t',\omega)\mathrm{d}t'. \tag{8.80}$$

From lemma 8.9, the right of (8.80) vanishes for $j = 0,1,2,3$, yielding the $i = 1$ case of (8.76) is valid. Repeating this argument proves the statement for $i = 2,3$ in the case $\mathbf{a} = 0$.

For the general case, we use induction on $|\mathbf{a}|$. Let m be a positive integer, and assume the statement has been proved for multiindices of length $|\mathbf{a}| \le m-1$. Apply $\not{D}^{\mathbf{a}}$ to both sides of the Teukolsky equation, and Taylor expand the result at $\mathscr{I}^+$. This yields a version of system (8.77) for $\not{D}^{\mathbf{a}} A_i$, $i = 0,\dots,3$, which again has the form

$$\partial_t \not{D}^{\mathbf{a}} A_i = \mathbf{L}_i{}^k \not{D}^{\mathbf{a}} A_k + [\not{D}^{\mathbf{a}}, \mathbf{L}_i{}^k] A_k. \tag{8.81}$$

The last term in (8.81) can be expressed in terms of $\not{D}^{\mathbf{b}} A_k$ with $|\mathbf{b}| \le m-1$ and $k < i$. This means that we can argue as above and use the fact that the system $\mathbf{L}_i{}^k$ is strictly lower triangular, to get the statement for $|\mathbf{a}| = m$. This completes the proof of the lemma. ☐

8.3. **Expansion for the spin-weight -2 Teukolsky scalar.**

Lemma 8.11 (Control of the Teukolsky scalar at and prior to Σ_{t_0})**.** *Let* $\hat{\psi}_{-2}$ *be a scalar of spin-weight* -2 *and* $\{\hat{\psi}_{-2}^{(i)}\}_{i=0}^4$ *be as in definition 6.1. There is a regularity constant* K *such that the following holds. Let* $j,k \in \mathbb{N}$ *such that* $k - j - K$ *is sufficiently large. Assume the BEAM condition from definition 6.8 holds. Let* $\mathbb{I}_{-2}^k$ *be as in definition 6.9, and let* $\mathbb{I}_{\text{init}}^{k;9}(\hat{\psi}_{-2})$ *be as in definition 4.20.*

(1)

$$\mathbb{I}^{k-4}_{-2} + \sum_{i=0}^{4} \|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k-4}_{-1-}(\Omega^{\text{early}}_{\text{init},t_0})} \lesssim \mathbb{I}^{k;9}_{\text{init}}(\hat{\psi}_{-2}). \tag{8.82}$$

(2) For $i \in \{0, \dots, 4\}$, and for any $t \le t_0$,

$$\int_{S^2_{t,\infty}} |\mathcal{L}^j_\xi \hat{\psi}^{(i)}_{-2}|^2_{k-j-K,\not{D}} \mathrm{d}^2\mu \lesssim \langle t \rangle^{-9+2i-2j+} \|\hat{\psi}_{-2}\|^2_{H^k_9(\Sigma_{\text{init}})}. \tag{8.83}$$

Proof. Consider estimating norms on Σ_t by those on Σ_{init} for $t \le t_0$. The basic estimate on $\hat{\psi}^{(i)}_{-2}$ in lemma 6.10 can essentially be repeated. In particular, from lemma 5.7 on spin-weighted wave equations in the early region, from the 5-component system (6.3), and from the relation between $\hat{\varphi}^{(i)}_{-2}$ and $\hat{\psi}^{(i)}_{-2}$ norms in lemma 6.5, it follows that there is a constant $\bar{R}_0$ such that, for all $\alpha \in [\delta, 2-\delta]$, $R_0 \ge \bar{R}_0$, and $t \le t_0$,

$$\begin{aligned}
&\sum_{i=0}^{4} \left(\|rV\hat{\psi}^{(i)}_{-2}\|^2_{W^k_{\alpha-2}(\Sigma^{R_0}_t)} + \|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k+1}_{-2}(\Sigma^{R_0}_t)} + \|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k+1}_{\alpha-3}(\Omega^{\text{early},R_0}_{\text{init},t})} \right) \\
&\lesssim \sum_{i=0}^{4} \|\hat{\psi}^{(i)}_{-2}\|^2_{H^{k+1}_{\alpha-1}(\Sigma_{\text{init}})} \\
&\quad + \sum_{i=0}^{4} \|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k+1}_0(\Omega^{\text{early},R_0-M,R_0}_{\text{init},t})} + \sum_{i=0}^{4} \|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k+1}_{-\delta}(\Sigma^{R_0-M,R_0}_t)} \\
&\quad + \sum_{i=0}^{4} \|Mr^{-1}\hat{\psi}^{(i)}_{-2}\|^2_{W^{k+1}_{\alpha-3}(\Omega^{\text{early},R_0-M}_{\text{init},t})}.
\end{aligned} \tag{8.84}$$

To treat the last term on the right-hand side, note that we can take R_0 sufficiently large such that $\|Mr^{-1}\hat{\psi}^{(i)}_{-2}\|^2_{W^{k+1}_{\alpha-3}(\Omega^{\text{early},R_0}_{\text{init},t})}$ can be absorbed into the $\|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k+1}_{\alpha-3}(\Omega^{\text{early},R_0}_{\text{init},t})}$ terms on the left, leaving $\|Mr^{-1}\hat{\psi}^{(i)}_{-2}\|^2_{W^{k+1}_{\alpha-3}(\Omega^{\text{early},R_0-M,R_0}_{\text{init},t})}$. For all $t \le t_0$, since R_0 is bounded, the terms

$$\|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k+1}_{\alpha-2}(\Sigma^{R_0-M,R_0}_t)}, \quad \|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k+1}_{\alpha-3}(\Omega^{\text{early},R_0-M,R_0}_{\text{init},t})} \tag{8.85}$$

can be bounded by a multiple of the initial norm

$$\|\hat{\psi}^{(i)}_{-2}\|^2_{H^{k+1}_{\alpha-1}(\Sigma_{\text{init}})}, \tag{8.86}$$

by standard exponential growth estimates for wave-like equations. Similarly, since $\Omega^{\text{early}}_{\text{init},t} \cap \{r \le R_0\}$ is bounded in spacetime, standard exponential growth estimates can be used to bound the energy on the upper boundary. Thus,

$$\begin{aligned}
&\sum_{i=0}^{4} \left(\|rV\hat{\psi}^{(i)}_{-2}\|^2_{W^k_{\alpha-2}(\Sigma_t)} + \|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k+1}_{-2}(\Sigma_t)} + \|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k+1}_{\alpha-3}(\Omega^{\text{early}}_{\text{init},t})} \right) \\
&\lesssim \sum_{i=0}^{4} \|\hat{\psi}^{(i)}_{-2}\|^2_{H^{k+1}_{\alpha-1}(\Sigma_{\text{init}})}.
\end{aligned} \tag{8.87}$$

In particular, with $\alpha = 2-\delta$, and recalling the $\hat{\psi}^{(i)}_{-2}$ are related via derivatives with an r^2 weight, but the norms $\|\varphi\|^2_{H^k_\alpha(\Sigma_{\text{init}})}$ are based on an r^1 weight for each derivative, (8.87) for $t = t_0$ yields

$$\mathbb{I}^{k+1}_{-2} + \sum_{i=0}^{4} \|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k+1}_{-1-\delta}(\Omega^{\text{early}}_{\text{init},t_0})} \lesssim \sum_{i=0}^{4} \|\hat{\psi}^{(i)}_{-2}\|^2_{H^{k+1}_{1-\delta}(\Sigma_{\text{init}})} \lesssim \|\hat{\psi}_{-2}\|^2_{H^{k+5}_{9-\delta}(\Sigma_{\text{init}})}. \tag{8.88}$$

Reindexing and using the notation introduced in definition 4.20 gives (8.82).

For t very negative, it is possible to prove stronger estimates by combining the ideas in the proofs of decay for the spin-weight -2 Teukolsky scalar in section 6 and of decay for the spin-weight $+2$ Teukolsky scalar as $t \to -\infty$ in the proof of theorem 7.8. In particular, for $i \in \{2, 3, 4\}$, one follows the proof of lemma 6.11, and, for $i \in \{0,1\}$, one follows that of theorem 6.13 in the case of the exterior region.

Let $\|\hat{\psi}_{-2}^{(i)}\|^2_{F^k(\mathscr{I}^+_{-\infty,t})}$ be as in definition 4.22. As in the proof of theorem 7.8, let $r(t)$ denote the value of r corresponding to the intersection of Σ_{init} and Σ_t, and recall that, for R_0 fixed and t sufficiently negative, we have that $r(t) > R_0$ and $r(t) \sim -t$. Recall that the proof of (8.84) and (8.87) is based on lemma 5.7 and in particular an application of Stokes' theorem, and hence we may add a term of the form

$$\sum_{i=0}^{4} \|\hat{\psi}_{-2}^{(i)}\|^2_{F^k(\mathscr{I}^+_{-\infty,t})} \tag{8.89}$$

on the left of (8.87) and replace Σ_{init} by $\Sigma_{\text{init}}^{r(t)} = \Sigma_{\text{init}} \cap \{r > r(t)\}$. Further, the resulting inequality holds with the summation over $i \in \{0,\dots,4\}$ replaced by summation over $i' \in \{0,\dots,i\}$ for $i \in \{2,3,4\}$. We now have, for $\alpha \in [\delta, 2-\delta]$,

$$\begin{aligned}\sum_{i'=0}^{i} \|\hat{\psi}_{-2}^{(i')}\|^2_{F^k(\mathscr{I}^+_{-\infty,t})} &\lesssim \sum_{i'=0}^{i} \|\hat{\psi}_{-2}^{(i')}\|^2_{H^{k+1}_{\alpha-1}(\Sigma_{\text{init}}^{r(t)})} \\ &\lesssim \|\hat{\psi}_{-2}\|^2_{H^{k+1+i}_{\alpha+2i-1}(\Sigma_{\text{init}}^{r(t)})}.\end{aligned} \tag{8.90}$$

With $\alpha = 2-\delta$ and $i = 4$, we get the weight $9-\delta$ as in (8.82). On the other hand, taking $\alpha = \delta$, one finds, for $i \in \{2,3,4\}$,

$$\begin{aligned}\sum_{i'=0}^{i} \|\hat{\psi}_{-2}^{(i')}\|^2_{F^k(\mathscr{I}^+_{-\infty,t})} &\lesssim \|\hat{\psi}_{-2}\|^2_{H^{k+1+i}_{\delta+2i-1}(\Sigma_{\text{init}}\cap\{r>r(t)\})} \\ &\lesssim r(t)^{-10+2i+2\delta} \|\hat{\psi}_{-2}\|^2_{H^{k+1+i}_{9-\delta}(\Sigma_{\text{init}}\cap\{r>r(t)\})} \\ &\lesssim |t|^{-10+2i+2\delta} \|\hat{\psi}_{-2}\|^2_{H^{k+1+i}_{9-\delta}(\Sigma_{\text{init}})}.\end{aligned} \tag{8.91}$$

The case $i = 2$ of (8.91) gives, after renaming i' to i, the estimate

$$\|\hat{\psi}_{-2}^{(i)}\|^2_{F^k(\mathscr{I}^+_{-\infty,t})} \lesssim |t|^{-6+2\delta} \|\hat{\psi}_{-2}\|^2_{H^{k+3}_{9-\delta}(\Sigma_{\text{init}})}, \quad \text{for } i \in \{0,1,2\}. \tag{8.92}$$

Since $\mathcal{L}_\xi$ commutes with the Teukolsky equation, but each derivative is weighted with r^{-1} in L^2 in the definition of $\|\hat{\psi}_{-2}\|_{H^{k+1+i}_{9-\delta}(\Sigma_{\text{init}})}$, one obtains an improved decay rate for $\mathcal{L}_\xi^j \hat{\psi}_{-2}^{(i)}$,

$$\begin{aligned}\sum_{i=0}^{2} \|\mathcal{L}_\xi^j \hat{\psi}_{-2}^{(i)}\|^2_{F^k(\mathscr{I}^+_{-\infty,t})} &\lesssim r(t)^{-6+2\delta} \|\mathcal{L}_\xi^j \hat{\psi}_{-2}\|^2_{H^{k+3}_{9-\delta}(\Sigma_{\text{init}}\cap\{r>r(t)\})} \\ &\lesssim |t|^{-6-2j+2\delta} \|\hat{\psi}_{-2}\|^2_{H^{k+j+3}_{9-\delta}(\Sigma_{\text{init}})}, \quad \text{for } j \in \mathbb{N}.\end{aligned} \tag{8.93}$$

Restricting the system (6.43) to $\mathscr{I}^+$, using (6.44), and taking $R = 0$, one finds a system of the form

$$L \begin{pmatrix} \hat{\psi}_{-2}^{(0)} \\ \hat{\psi}_{-2}^{(1)} \end{pmatrix} = A\mathcal{L}_\xi \begin{pmatrix} \hat{\psi}_{-2}^{(0)} \\ \hat{\psi}_{-2}^{(1)} \end{pmatrix} + B \begin{pmatrix} 0 \\ \mathcal{L}_\xi \hat{\psi}_{-2}^{(2)} \end{pmatrix}, \tag{8.94}$$

with

$$L = \begin{pmatrix} L^{(0)} & 0 \\ 6 + 6\frac{a}{M}\mathcal{L}_\eta & L^{(1)} \end{pmatrix} \tag{8.95}$$

and

$$L^{(i)} = 2\mathring{ð}\mathring{ð}' - 2(2+i), \quad i = 0,1 \tag{8.96}$$

and where A is a matrix of operators of maximal order 1 involving $\mathcal{L}_\xi, \mathcal{L}_\eta$, and constants, and with bounded, t-independent coefficients, and where $B \in \mathbb{R}$. In particular, the first row of system (8.94) is of the form

$$L^{(0)}\hat{\psi}_{-2}^{(0)} = A^{(00)}\mathcal{L}_\xi \hat{\psi}_{-2}^{(0)} + A^{(01)}\mathcal{L}_\xi \hat{\psi}_{-2}^{(1)} \tag{8.97}$$

where $A^{(00)}, A^{(01)}$ are operators of maximal order 1 involving $\mathcal{L}_\xi$, $\mathcal{L}_\eta$ and constants, with bounded t-independent coefficients.

The operators $L^{(i)}$, $i = 0, 1$ are invertible on Sobolev spaces on the cross-sections $S^2_\tau = \mathscr{I}^+ \cap \{t = \tau\}$. Thus, since L is lower triangular and the off-diagonal term has maximal order 1 with the first order part involving only derivatives tangent to S_t, we find that L is invertible on Sobolev spaces on S^2_t. In particular, we have

$$\left\| \begin{pmatrix} \hat{\psi}^{(0)}_{-2} \\ \hat{\psi}^{(1)}_{-2} \end{pmatrix} \right\|_{L^2(S^2_t)} \lesssim \left\| L \begin{pmatrix} \hat{\psi}^{(0)}_{-2} \\ \hat{\psi}^{(1)}_{-2} \end{pmatrix} \right\|_{L^2(S^2_t)} . \tag{8.98}$$

This estimate yields corresponding estimates for the semi-norms $F^k(\mathscr{I}^+_{-\infty,t})$. Using (8.93) and (8.94), this gives

$$\|\mathcal{L}^j_\xi \hat{\psi}^{(i)}_{-2}\|^2_{F^{k-j-K}(\mathscr{I}^+_{-\infty,t})} \lesssim |t|^{-8-2j+2\delta} \|\hat{\psi}_{-2}\|^2_{H^k_{9-\delta}(\Sigma_{\text{init}})}, \quad \text{for } i \in \{0,1\},\ j \in \mathbb{N}. \tag{8.99}$$

Finally, using (8.97) and (8.99) gives

$$\|\mathcal{L}^j_\xi \hat{\psi}^{(0)}_{-2}\|^2_{F^{k-j-K}(\mathscr{I}^+_{-\infty,t})} \lesssim |t|^{-10-2j+2\delta} \|\hat{\psi}_{-2}\|^2_{H^k_{9-\delta}(\Sigma_{\text{init}})}, \quad \text{for } j \in \mathbb{N}. \tag{8.100}$$

In particular, with $j = 0$, we have

$$\|\hat{\psi}^{(0)}_{-2}\|^2_{F^{k-K}(\mathscr{I}^+_{-\infty,t})} \lesssim |t|^{-10+2\delta} \|\hat{\psi}_{-2}\|^2_{H^k_{9-\delta}(\Sigma_{\text{init}})}. \tag{8.101}$$

Reindexing and using $\hat{\psi}^{(0)}_{-2} = \hat{\psi}_{-2}$ gives

$$\|\mathcal{L}^j_\xi \hat{\psi}_{-2}\|^2_{F^{k-j-K}(\mathscr{I}^+_{-\infty,t})} \lesssim |t|^{-10-2j+2\delta} \|\hat{\psi}_{-2}\|^2_{H^k_{9-\delta}(\Sigma_{\text{init}})}. \tag{8.102}$$

From the fundamental theorem of calculus, the Cauchy-Schwarz inequality, and the definition of the F^k norm, one finds

$$\begin{aligned} \|\mathcal{L}^{j+1}_\xi \hat{\psi}^{(i)}_{-2}\|^2_{W^{k-j-K}(S^2)} &\lesssim \int_{\mathscr{I}^+_{-\infty,t}} |\mathcal{L}^{j+1}_\xi \hat{\psi}^{(i)}_{-2}|_{k-j-K,\mathbb{S}} |\mathcal{L}^{j+2}_\xi \hat{\psi}^{(i)}_{-2}|_{k-j-K,\mathbb{S}} \mathrm{d}^3\mu \\ &\lesssim \|\mathcal{L}^j_\xi \hat{\psi}^{(i)}_{-2}\|_{F^{k-j-K}(\mathscr{I}^+_{-\infty,t})} \|\mathcal{L}^{j+1}_\xi \hat{\psi}^{(i)}_{-2}\|_{F^{k-j-K}(\mathscr{I}^+_{-\infty,t})} \\ &\lesssim |t|^{-11-2j+2i+2\delta} \|\hat{\psi}^{(i)}_{-2}\|^2_{H^k_{9-\delta}(\Sigma_{\text{init}})}. \end{aligned} \tag{8.103}$$

Taking the square root and applying the fundamental theorem of calculus again, one finds

$$\|\mathcal{L}^j_\xi \hat{\psi}^{(i)}_{-2}\|_{W^{k-j-K}(S^2)} \lesssim |t|^{-9/2-j+i+\delta} \|\hat{\psi}^{(i)}_{-2}\|_{H^k_{9-\delta}(\Sigma_{\text{init}})}, \tag{8.104}$$

which gives (8.83). □

Lemma 8.12 (The Teukolsky scalar $\hat{\psi}_{-2}$ has an expansion)**.** *Let $\hat{\psi}_{-2}$ be a scalar of spin-weight -2 and $\{\hat{\psi}^{(i)}_{-2}\}^4_{i=0}$ be as in definition 6.1. Assume $\hat{\psi}_{-2}$ satisfies the Teukolsky equation (3.25a). Let $\hat{\psi}_{+2}$ be a spin-weight $+2$ scalar that is a solution of the Teukolsky equation (3.25b). There is a regularity constant K such that the following holds. Assume the BEAM condition from definition 6.8 holds. Assume the pointwise condition (3) from definition 7.1 holds. Let $k \in \mathbb{N}$ such that $k - K$ is sufficiently large, and let $\alpha_1[\hat{\psi}_{-2}] = 10-$.*

Then $\hat{\psi}_{-2}$ has a $(k-K, 3, 0, \alpha_1[\hat{\psi}_{-2}], D^2)$ expansion where $D^2 = \mathbb{I}^{k;9}_{\text{init}}(\hat{\psi}_{-2})$.

Proof. Throughout the proof, K denotes a regularity constant that may vary from line to line. Before considering $\hat{\psi}_{-2}$, for a general spin-weighted scalar φ, consider Taylor expansions in the R variable. Recall Taylor's expansion lemma 4.35. In particular, consider $n \in \mathbb{N}$, $A > 0$, $f = f(R) \in C^{n+1}([-\epsilon, A])$ for some $\epsilon > 0$, and P_n the order n Taylor polynomial in R about $R = 0$. Observe that $\mathrm{d}R = -r^{-2}\mathrm{d}r$, so if the L^2 norms are defined in terms of $\mathrm{d}r$, we get from Taylor's expansion lemma 4.35 that for $-1 < \beta < 1$, the following Taylor remainder estimate:

$$\|r^{n+\beta/2}(f - P_n)\|_{L^2((1/A,\infty))} \lesssim_{n,\beta} \|r^{\beta/2-1} f^{(n+1)}\|_{L^2((1/A,\infty))}. \tag{8.105}$$

Consider now a spin-weighted scalar φ defined in the Kerr exterior and $j \in \mathbb{N}$. The Taylor remainder estimate implies that if there is the expansion

$$\varphi = \sum_{i=0}^{j} \frac{R^i}{i!} \varphi_i(t,\omega) + \varphi_{\mathrm{rem};j}, \tag{8.106}$$

we get, for $-1 < \beta < 1$,

$$\begin{aligned} \|\varphi_{\mathrm{rem};j}\|_{W^0_{2j+\beta}(\Omega^{\mathrm{ext}}_{t,\infty})} &= \|r^{j+\beta/2}\varphi_{\mathrm{rem};j}\|_{W^0_0(\Omega^{\mathrm{ext}}_{t,\infty})} \\ &\lesssim_{j,\beta} \|r^{\beta/2-1}(\partial_R)^{j+1}\varphi\|_{W^0_0(\Omega^{\mathrm{ext}}_{t,\infty})} \\ &\lesssim_{j,\beta} \|(\partial_R)^{j+1}\varphi\|_{W^0_{\beta-2}(\Omega^{\mathrm{ext}}_{t,\infty})}. \end{aligned} \tag{8.107}$$

Substituting $\beta = \alpha - 1$, one finds, for $0 < \alpha < 2$,

$$\|\varphi_{\mathrm{rem};j}\|_{W^0_{2j-1+\alpha}(\Omega^{\mathrm{ext}}_{t,\infty})} \lesssim_{j,\alpha} \|(\partial_R)^{j+1}\varphi\|_{W^0_{\alpha-3}(\Omega^{\mathrm{ext}}_{t,\infty})}. \tag{8.108}$$

Commuting with the $\mathbb{D}$ operators only introduces lower-order terms, so that, for $0 < \alpha < 2$ and $k \in \mathbb{N}$,

$$\|\varphi_{\mathrm{rem};j}\|_{W^k_{2j-1+\alpha}(\Omega^{\mathrm{ext}}_{t,\infty})} \lesssim_{j,\alpha} \|(\partial_R)^{j+1}\varphi\|_{W^k_{\alpha-3}(\Omega^{\mathrm{ext}}_{t,\infty})}. \tag{8.109}$$

Arguing in a similar way, we have for $0 < \alpha < 2$ and $k \in \mathbb{N}$,

$$\|\varphi_{\mathrm{rem};j}\|_{W^k_{2j-1+\alpha}(\Omega^{\mathrm{early}}_{\mathrm{init},t_0})} \lesssim_{j,\alpha} \|(\partial_R)^{j+1}\varphi\|_{W^k_{\alpha-3}(\Omega^{\mathrm{early}}_{\mathrm{init},t_0})}. \tag{8.110}$$

Now, consider the existence of an expansion for $\hat{\psi}_{-2}$. From the expansion of ∂_R in (4.8) and the expansion of $\mathcal{L}_\xi$ in (2.35), for $r \geq 10M$ and $i \in \{0,\dots,4\}$, one has, with $\mathbb{X} = \{MY, \mathcal{L}_\eta\}$,

$$|\partial_R^i \hat{\psi}_{-2}| \lesssim \sum_{l=0}^{i} \sum_{|\mathbf{a}|\leq i-l} M^i O_\infty(1) |\mathbb{X}^{\mathbf{a}} \hat{\psi}^{(l)}_{-2}|. \tag{8.111}$$

Since for bounded r, in particular for $r \in [r_+, 10M]$, one has ∂_R is in the span of Y, V, $\mathcal{L}_\eta$, one finds that, for all r, equation (8.111) remains valid. Since the $\partial_R^i \hat{\psi}_{-2}$ exist for $i \in \{0,\dots,4\}$, the following Taylor expansion exists

$$\hat{\psi}_{-2} = \sum_{i=0}^{3} \frac{R^i}{i!} (\hat{\psi}_{-2})_i + (\hat{\psi}_{-2})_{\mathrm{rem};3}. \tag{8.112}$$

This proves condition (8.3a) in the definition of the expansion.

Using estimate (6.38) in theorem 6.13 and estimate (8.82) in lemma 8.11, and taking $\alpha = 2 - \delta \in (0,2)$, one finds for any $t \geq t_0$,

$$\|(\partial_R)^4 \hat{\psi}_{-2}\|^2_{W^{k-K}_{-1-\delta}(\Omega^{\mathrm{ext}}_{t,\infty})} \lesssim \sum_{i=0}^{4} \|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k-K}_{-1-\delta}(\Omega^{\mathrm{ext}}_{t,\infty})} \lesssim \mathbb{I}^{k-4}_{-2} \lesssim \mathbb{I}^{k;9}_{\mathrm{init}}(\hat{\psi}_{-2}). \tag{8.113}$$

Therefore, from the Taylor remainder bound (8.108), we conclude, for any $t \geq t_0$,

$$\|(\hat{\psi}_{-2})_{\mathrm{rem};3}\|^2_{W^{k-K}_{7-\delta}(\Omega^{\mathrm{ext}}_{t,\infty})} \lesssim \|(\partial_R)^4 \hat{\psi}_{-2}\|^2_{W^{k-K}_{-1-\delta}(\Omega^{\mathrm{ext}}_{t,\infty})} \lesssim \mathbb{I}^{k;9}_{\mathrm{init}}(\hat{\psi}_{-2}), \tag{8.114}$$

from which it follows that, letting

$$\alpha_1[\hat{\psi}_{-2}] = 10 - 11\delta \tag{8.115}$$

and noting that $(10 - 11\delta) - 3 < 7 - \delta$, we have, for any $t \geq t_0$,

$$\|(\hat{\psi}_{-2})_{\mathrm{rem};3}\|^2_{W^{k-K}_{\alpha_1[\hat{\psi}_{-2}]-3}(\Omega^{\mathrm{ext}}_{t,\infty})} \lesssim \mathbb{I}^{k;9}_{\mathrm{init}}(\hat{\psi}_{-2}), \tag{8.116}$$

which proves the remainder condition (8.3b) in the definition of an expansion.

In the region $\Omega^{\text{early}}_{\text{init},t_0}$, using the bound (8.110), using the estimates (8.110) and (8.82), and taking $\alpha = 2-\delta$, we conclude

$$\begin{aligned}
\|(\hat{\psi}_{-2})_{\text{rem};3}\|^2_{W^{k-K}_{7-\delta}(\Omega^{\text{early}}_{\text{init},t_0})} &\lesssim \|(\partial_R)^4\hat{\psi}_{-2}\|^2_{W^{k-K}_{-1-\delta}(\Omega^{\text{early}}_{\text{init},t_0})} \\
&\lesssim \sum_{i=0}^{4}\|\hat{\psi}^{(i)}_{-2}\|^2_{W^{k-K}_{-1-\delta}(\Omega^{\text{early}}_{\text{init},t_0})} \\
&\lesssim \mathbb{I}^{k;9}_{\text{init}}(\hat{\psi}_{-2}). \qquad (8.117)
\end{aligned}$$

Hence, by letting $\alpha_1[\hat{\psi}_{-2}]$ be as in equation (8.115) and noting $(10-11\delta)-3 < 7-\delta$, it follows that

$$\|(\hat{\psi}_{-2})_{\text{rem};3}\|^2_{W^{k-K}_{\alpha_1[\hat{\psi}_{-2}]-3}(\Omega^{\text{early}}_{\text{init},t_0})} \lesssim \mathbb{I}^{k;9}_{\text{init}}(\hat{\psi}_{-2}), \qquad (8.118)$$

and this proves condition (8.3c) in the definition of an expansion.

Consider the Taylor expansion terms $(\hat{\psi}_{-2})_i$ for $i \in \{0,\dots,3\}$. From the pointwise bound on the $\hat{\psi}^{(i)}_{-2}$ in inequality (6.39), one finds that there are the decay estimates, for $i \in \{0,\dots,3\}$ and $t \geq t_0$,

$$\int_{S^2} |(\hat{\psi}_{-2})_i(t,\omega)|^2_{k-K,\not{D}}\,\mathrm{d}^2\mu \lesssim t^{-9+2i+11\delta}\mathbb{I}^{k;9}_{\text{init}}(\hat{\psi}_{-2}). \qquad (8.119)$$

For $t \geq t_0$, this proves that there are the decay estimates for the expansion terms, which is condition (8.3d). For $t \leq t_0$, the same argument applies using the decay estimate in equation (8.83). Thus, condition (8.3d) holds for all $t \in \mathbb{R}$.

From the assumptions on $\hat{\psi}_{+2}$ and $\hat{\psi}_{-2}$, lemmas 8.9 and 8.10 can be applied, and equation (8.76) states the vanishing of the integral along $\mathscr{I}^+$ of the $\hat{\psi}^{(i)}_{-2}$. From the relation between the $\hat{\psi}^{(i)}_{-2}$ and the $(\hat{\psi}_{-2})_i$ in equation (8.111), it follows that the integral of the $(\hat{\psi}_{-2})_i$ along $\mathscr{I}^+$ vanishes. Thus, condition (8.3e) holds, and $\hat{\psi}_{-2}$ has the desired expansion. □

8.4. **Estimates in the exterior region.** This section proves decay estimates for $\hat{\sigma}'$, $\widehat{G}_2$, $\hat{\tau}'$, $\widehat{G}_1$, $\hat{\beta}'$, and $\widehat{G}_0$ in the exterior region $r \geq t$. The proof consists of three major components. First, $\hat{\psi}_{-2}$ has an expansion by lemma 8.12. Second, the scalars $\hat{\sigma}', \widehat{G}_2, \hat{\tau}', \widehat{G}_1, \hat{\beta}', \widehat{G}_0$ are related to each other by the transport equations in the first-order formulation of the Einstein equation in lemma 3.8. Third, lemma 8.6 states that if the source for a transport equation has an expansion, then the solution has an expansion and satisfies decay estimates. Thus, iterating through $\hat{\sigma}'$, $\widehat{G}_2$, $\hat{\tau}'$, $\widehat{G}_1$, $\hat{\beta}'$, and $\widehat{G}_0$ one finds each of these has an expansion and decays.

The following indices are useful in this iteration process. These indices are such that, for φ, $s[\varphi]$ is the spin weight of φ, and $l[\varphi]$ and $m[\varphi]$ will be the l and m arguments in the expansion of φ.

Definition 8.13. Define

$$\begin{aligned}
s[\hat{\psi}_{-2}] &= -2, & l[\hat{\psi}_{-2}] &= 3, & m[\hat{\psi}_{-2}] &= 0, && (8.120a)\\
s[\hat{\sigma}'] &= -2, & l[\hat{\sigma}'] &= 2, & m[\hat{\sigma}'] &= 0, && (8.120b)\\
s[\widehat{G}_2] &= -2, & l[\widehat{G}_2] &= 1, & m[\widehat{G}_2] &= 0, && (8.120c)\\
s[\hat{\tau}'] &= -1, & l[\hat{\tau}'] &= 3, & m[\hat{\tau}'] &= 2, && (8.120d)\\
s[\widehat{G}_1] &= -1, & l[\widehat{G}_1] &= 0, & m[\widehat{G}_1] &= 0, && (8.120e)\\
s[\hat{\beta}'] &= -1, & l[\hat{\beta}'] &= 2, & m[\hat{\beta}'] &= 3, && (8.120f)\\
s[\widehat{G}_0] &= 0, & l[\widehat{G}_0] &= 1, & m[\widehat{G}_0] &= 2. && (8.120g)
\end{aligned}$$

For all $\beta \in \mathbb{R}$ we also define $s[M^\beta\varphi] = s[\varphi], l[M^\beta\varphi] = l[\varphi], m[M^\beta\varphi] = m[\varphi]$.

Definition 8.14 (Initial data norms)**.** Define the following set of dimensionless fields

$$\Phi = \{M\hat{\psi}_{-2}, \hat{\sigma}', M^{-1}\widehat{G}_2, M\hat{\tau}', M^{-2}\widehat{G}_1, \hat{\beta}', M^{-1}\widehat{G}_0\}. \qquad (8.121)$$

For any $k \in \mathbb{N}$, define

$$\mathbb{I}^k_{\mathrm{init}}[\Phi] = \sum_{\varphi\in\Phi} \mathbb{I}^{k;2l[\varphi]+3}_{\mathrm{init}}(\varphi). \tag{8.122}$$

Lemma 8.15 (Exterior estimates)**.** *Consider an outgoing BEAM solution of the linearized Einstein equation satisfying as in definition 8.2. There is a regularity constant K such that the following hold. Let $k \in \mathbb{N}$ such that $k-K$ is sufficiently large. For $\varphi \in \Phi$,*

(1) φ has a $(k-K, l[\varphi], m[\varphi], 2l[\varphi]+4-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$ expansion,
(2) for $q \in \{0,1\}$ and $t \geq t_0$,

$$\|\mathcal{L}^q_\xi \varphi\|^2_{W^{k-K-q}_{2l[\varphi]+2-}(\Xi_{t,\infty})} \lesssim t^{-2q}\mathbb{I}^k_{\mathrm{init}}[\Phi], \tag{8.123}$$

(3) and, for $t \geq t_0$ and $r \geq t$,

$$|\varphi|^2_{k-K,\mathbb{D}} \lesssim r^{-2m[\varphi]} t^{2m[\varphi]-3-2l[\varphi]+}\mathbb{I}^k_{\mathrm{init}}[\Phi]. \tag{8.124}$$

Proof. For ease of presentation we will here use mass normalization as in definition 4.4, and throughout this proof, K denotes a regularity constant that may vary from line to line. The overall strategy of this proof is to use the expansion for $\hat\psi_{-2}$ from lemma 8.12, the hierarchy of transport equations (3.13), lemma 8.6 to conclude that solutions of the transport equation have expansions if the source does, and lemma 8.7 when a transformation of the expansions for the source is required. The details now follow.

From lemma 8.12, $\hat\psi_{-2}$ has a $(k-K, 3, 0, 10-, D[\hat\psi_{-2}]^2)$ expansion where

$$D[\hat\psi_{-2}]^2 = \mathbb{I}^{k;9}_{\mathrm{init}}(\hat\psi_{-2}) \leq \mathbb{I}^k_{\mathrm{init}}[\Phi]. \tag{8.125}$$

The decay estimates of the transition flux for $\varphi = \hat\psi_{-2}$ follow from integrating the pointwise estimates (6.39) on $\Xi_{t,\infty}$ and making use of (8.82), while the pointwise decay estimates (8.124) for $\varphi = \hat\psi_{-2}$ follow directly from the estimates (6.39) and (8.82).

Consider $\hat\sigma'$. The transport equation (3.13a) states that

$$Y(\hat\sigma') = -\frac{12\bar\kappa_{1'}\hat\psi_{-2}}{\sqrt{r^2+a^2}}. \tag{8.126}$$

The factor $\frac{\bar\kappa_{1'}}{\sqrt{r^2+a^2}}$ is a rational function in $\bar\kappa_{1'}/\sqrt{r^2+a^2}$ of homogeneous degree 0. Thus, lemma 8.7 implies that $-\frac{12\bar\kappa_{1'}\hat\psi_{-2}}{\sqrt{r^2+a^2}}$ also has a $(k-K, 3, 0, 10-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$ expansion. Thus, lemma 8.6 implies that $\hat\sigma'$ has a $(k-K, 3-1, 0, (10-2)-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$ expansion, that is a $(k-K, 2, 0, 8-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$ expansion.

Consider $\widehat{G}_2$. The transport equation (3.13b) states

$$Y(\widehat{G}_2) = -\tfrac{2}{3}\hat\sigma'. \tag{8.127}$$

From this, lemma 8.6 implies that $\widehat{G}_2$ has a $(k-K, 2-1, 0, (8-2)-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$ expansion, that is a $(k-K, 1, 0, 6-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$ expansion.

Consider $\hat\tau'$. The transport equation (3.13c) states

$$\begin{aligned} Y(\hat\tau') &= -\frac{\kappa_1(\eth - 2\tau + 2\bar\tau')\hat\sigma'}{6\bar\kappa_{1'}{}^2} \\ &= \frac{1}{6\bar\kappa_{1'}^2}\left(\kappa_1 \eth\, \hat\sigma' - 2\tau\kappa_1\hat\sigma' + 2\bar\tau'\kappa_1\hat\sigma'\right). \end{aligned} \tag{8.128}$$

The operator $\kappa_1 \eth\, \hat\sigma'$ can be expanded in terms of $\mathbb{D}$ with rational coefficients of order at most 0. Thus, $\kappa_1 \eth\, \hat\sigma'$ has an expansion with indices $(k-K, 2, 0, 8-, \mathbb{I}^k_{\mathrm{init}}[\Phi]) = (k-K, 2, 0, 8-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$. The terms $-2\tau\kappa_1\hat\sigma'$ and $2\bar\tau'\kappa_1\hat\sigma'$ have similar expansions, where the regularity index can be trivially lowered to match that for $\kappa_1 \eth\, \hat\sigma'$. The coefficient $\bar\kappa_{1'}^{-2}$ has homogeneous degree -2. Thus, from the expansion for $\hat\sigma'$, one finds that the right-hand side of equation (8.128) has an expansion with indices $(k-K, 2+2, 0+2, 8+4-, \mathbb{I}^k_{\mathrm{init}}[\Phi]) = (k-K, 4, 2, 12+4-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$. Thus, $\hat\tau'$ has an expansion with indices $(k-K, 4-1, 2, (12-2)-, \mathbb{I}^k_{\mathrm{init}}[\Phi]) = (k-K, 3, 2, 10-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$.

Consider $\widehat{G}_1$. The transport equation (3.13d) states

$$Y(\widehat{G}_1) = \frac{2\kappa_1{}^2\overline{\kappa}_{1'}{}^2\hat{\tau}'}{r^2} + \frac{\kappa_1{}^2\overline{\kappa}_{1'}(\eth - \tau + \bar{\tau}')\widehat{G}_2}{2r^2}. \tag{8.129}$$

The term involving $\hat{\tau}'$ has an expansion with indices $(k-K, 3-2, 2-2, (10-4)-, \mathbb{I}^k_{\mathrm{init}}[\Phi]) = (k-K, 1, 0, 6-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$. The term with $\widehat{G}_2$ has an expansion with indices $(k-K, 1, 0, 6-, \mathbb{I}^k_{\mathrm{init}}[\Phi]) = (k-K, 1, 0, 6-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$. Thus, the right-hand side has an expansion with indices $(k-K, 1, 0, 6-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$, and $\widehat{G}_1$ has an expansion with indices $(k-K, 1-1, 0, (6-2)-, \mathbb{I}^k_{\mathrm{init}}[\Phi]) = (k-K, 0, 0, 4-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$.

Consider $\hat{\beta}'$. The transport equation (3.13e) states

$$Y(\hat{\beta}') = \frac{r\widehat{G}_1}{6\kappa_1{}^2\overline{\kappa}_{1'}{}^2} + \frac{\kappa_1\tau\widehat{G}_2}{6\overline{\kappa}_{1'}{}^2}. \tag{8.130}$$

The term involving $\widehat{G}_1$ has an expansion with indices $(k-K, 0+3, 0+3, 4+6-, \mathbb{I}^k_{\mathrm{init}}[\Phi]) = (k-K, 3, 3, 10-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$. The term involving $\widehat{G}_2$ has an expansion with indices $(k-K, 1+3, 0+3, 6+6-, \mathbb{I}^k_{\mathrm{init}}[\Phi]) = (k-K, 4, 3, 12-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$. The first of these is more restrictive. Thus, the right-hand side has a $(k-K, 3, 3, 10-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$ expansion, and $\hat{\beta}'$ has an expansion with indices $(k-K, 3-1, 3, (10-2)-, \mathbb{I}^k_{\mathrm{init}}[\Phi]) = (k-K, 2, 3, 8-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$.

Finally, consider $\widehat{G}_0$. The transport equation (3.13f) states

$$Y(\widehat{G}_0) = -\frac{(\eth - \tau)\widehat{G}_1}{3\kappa_1} - \frac{\tau\widehat{G}_1}{r} - \frac{\bar{\tau}\overline{\widehat{G}_1}}{r} + \frac{2\kappa_1{}^2\overline{\kappa}_{1'}(\eth - \bar{\tau}')\hat{\beta}'}{r^2} - \frac{(\eth' - \bar{\tau})\overline{\widehat{G}_1}}{3\overline{\kappa}_{1'}} + \frac{2\kappa_1\overline{\kappa}_{1'}{}^2(\eth' - \tau')\overline{\hat{\beta}'}}{r^2}. \tag{8.131}$$

Complex conjugation does not change the indices in an expansion. The terms involving $\widehat{G}_1$ have an additional level of regularity and coefficients with homogeneous degree -2, so that the expansion has indices $(k-K, 0+2, 0+2, 4+4-, \mathbb{I}^k_{\mathrm{init}}[\Phi]) = (k-K, 2, 2, 8-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$. The terms involving $\hat{\beta}'$ also have one derivative but coefficients with homogeneous degree 0, so that the expansion has indices $(k-K, 2, 3, 8-, \mathbb{I}^k_{\mathrm{init}}[\Phi]) = (k-K, 2, 3, 8-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$. Thus, the right-hand side has an expansion with indices $(k-K, 2, 2, 8-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$, and $\widehat{G}_0$ has an expansion with indices $(k-K, 2-1, 2, (8-2)-, \mathbb{I}^k_{\mathrm{init}}[\Phi]) = (k-K, 1, 2, 6-, \mathbb{I}^k_{\mathrm{init}}[\Phi])$.

For each of $\hat{\sigma}'$, $\hat{G}_2$, $\hat{\tau}'$, $\hat{G}_1$, $\hat{\beta}'$, $\hat{G}_0$, lemma 8.6 was applied to obtain the expansion. This lemma also gives estimates for the integral on $\Xi_{t,\infty}$ and for pointwise norms. The pointwise bound (8.23b) is stronger than the bound (8.23a), so in all cases, one can apply the bound (8.23a). (Because of this observation, it is not necessary to track which of the two bounds holds, although a carefully tracking of this would reveal that the bound (8.23b) never holds in this argument.) □

8.5. **Estimates in the interior region.**

Lemma 8.16. *Let φ and ϱ be spin-weighted scalars which solve*

$$Y\varphi = \varrho, \tag{8.132}$$

and let $0 \le \underline{\alpha} < \overline{\alpha}$ be given. Let $k \in \mathbb{Z}^+$. Let $D \ge 0$.

Assume that for all $t \ge t_0 + h(t_0)$, $\alpha \in [\underline{\alpha}, \overline{\alpha}]$, and $q \in \{0,1\}$, φ and ϱ satisfy

$$\|\mathcal{L}^q_\xi\varphi\|^2_{W^{k-q}_\alpha(\Xi_{t,\infty})} \lesssim D^2 t^{\alpha-\overline{\alpha}-2q}, \tag{8.133a}$$

$$\|\mathcal{L}^q_\xi\varrho\|^2_{W^{k-q}_{\alpha+1}(\Omega^{\mathrm{near}}_{t,\infty})} \lesssim D^2 t^{\alpha-\overline{\alpha}-2q}, \tag{8.133b}$$

then, for all $\underline{\alpha} \le \alpha \le \overline{\alpha}$, the following holds. For all $q \in \{0,1\}$ and $t \ge t_0 + h(t_0)$,

$$\|\mathcal{L}^q_\xi\varphi\|^2_{W^{k-q}_\alpha(\Sigma^{\mathrm{int}}_t)} + \|\mathcal{L}^q_\xi\varphi\|^2_{W^{k-q}_{\alpha-1}(\Omega^{\mathrm{near}}_{t,\infty})} \lesssim D^2 t^{\alpha-\overline{\alpha}-2q}, \tag{8.134a}$$

and, if $k \ge 4$, then for all $t \ge t_0 + h(t_0)$ and $(t, r, \omega) \in \Omega^{\mathrm{near}}_{t,\infty}$,

$$|\varphi(t, r, \omega)|_{k-4,\mathbb{D}} \lesssim D^2 r^{-\frac{\alpha}{2}} t^{-\frac{\overline{\alpha}+1-\alpha}{2}}. \tag{8.134b}$$

Remark 8.17. For $t \ge t_0 + h(t_0)$, φ is determined in $\Omega^{\mathrm{near}}_{t,\infty}$ by ϱ and $\varphi\big|_{\Xi_{t_{\mathcal{C}}(t),\infty}}$.

Proof. For ease of presentation we will here use mass normalization as in definition 4.4. For $t \geq t_0 + h(t_0)$, inequality (5.23) gives

$$
\begin{aligned}
&\|\mathcal{L}_\xi^q \varphi\|^2_{W^{k-q}_\alpha(\Sigma^{\text{int}}_t)} + \|\mathcal{L}_\xi^q \varphi\|^2_{W^{k-q}_{\alpha-1}(\Omega^{\text{near}}_{t,\infty})} \\
&\quad \lesssim \|\mathcal{L}_\xi^q \varphi\|^2_{W^{k-q}_\alpha(\Xi_{t_{\mathfrak{C}}(t),\infty})} + \|\mathcal{L}_\xi^q \varrho\|^2_{W^{k-q}_{\alpha+1}(\Omega^{\text{near}}_{t,\infty})} \\
&\quad \lesssim D^2 t^{\alpha-\overline{\alpha}-2q},
\end{aligned}
\tag{8.135}
$$

which proves (8.134a). Here we have used assumption (8.133) and $t_{\mathfrak{C}}(t) \sim t$.

The estimate (4.48) gives for $t > t_0 + h(t_0)$,

$$
|r^{\frac{\alpha}{2}}\varphi|^2_{k-3,\mathbb{D}} \lesssim \|r^{\frac{\alpha}{2}}\varphi\|^2_{W^k_{-1}(\Sigma^{\text{int}}_t)} \lesssim \|\varphi\|^2_{W^k_{-1+\alpha}(\Sigma^{\text{int}}_t)}. \tag{8.136}
$$

Since $-1+\alpha < \alpha$, we find, in view of (8.134a) that $r^{\frac{\alpha}{2}}\varphi$ tends to zero as $t \nearrow \infty$. We can therefore apply lemma 4.33 which gives

$$
\begin{aligned}
|r^{\frac{\alpha}{2}}\varphi|^2_{k-4,\mathbb{D}} &\lesssim \|r^{\frac{\alpha}{2}}\varphi\|_{W^{k-1}_{-1}(\Omega^{\text{int}}_{t,\infty})} \|r^{\frac{\alpha}{2}}\mathcal{L}_\xi\varphi\|_{W^{k-1}_{-1}(\Omega^{\text{int}}_{t,\infty})} \\
&\leq \|r^{\frac{\alpha}{2}}\varphi\|_{W^{k-1}_{-1}(\Omega^{\text{near}}_{t,\infty})} \|r^{\frac{\alpha}{2}}\mathcal{L}_\xi\varphi\|_{W^{k-1}_{-1}(\Omega^{\text{near}}_{t,\infty})} \\
&\lesssim \|\varphi\|_{W^{k-1}_{\alpha-1}(\Omega^{\text{near}}_{t,\infty})} \|\mathcal{L}_\xi\varphi\|_{W^{k-1}_{\alpha-1}(\Omega^{\text{near}}_{t,\infty})},
\end{aligned}
\tag{8.137}
$$

which using (8.134a) proves (8.134b). □

Lemma 8.18. *There is a regularity constant K such that the following holds. Consider an outgoing BEAM solution of the linearized Einstein equation as in definition 8.2, with regularity $k \in \mathbb{N}$ and $k - K$ sufficiently large. For $\varphi \in \Phi \backslash \{M\hat{\psi}_{-2}\}$, let $m[\varphi]$, $l[\varphi]$ be as in definition 8.13, and set*

$$
\underline{\alpha}[\varphi] = 2\max\{m[\varphi]-1, 0\}+, \qquad \overline{\alpha}[\varphi] = 2l[\varphi] + 2-. \tag{8.138}
$$

The following hold for $\varphi \in \Phi \backslash \{M\hat{\psi}_{-2}\}$ and $t \geq t_0$.

(1) For $q \in \{0,1\}$ and $\alpha \in [\underline{\alpha}[\varphi], \overline{\alpha}[\varphi]]$, there are energy and Morawetz estimates in the interior region

$$
\|\mathcal{L}_\xi^q\varphi\|^2_{W^{k-K-q}_\alpha(\Sigma^{\text{int}}_t)} \lesssim \mathbb{I}^k_{\text{init}}[\Phi] t^{\alpha-\overline{\alpha}[\varphi]-2q}, \tag{8.139a}
$$

$$
\|\mathcal{L}_\xi^q\varphi\|^2_{W^{k-K-q}_{\alpha-1}(\Omega^{\text{near}}_{t,\infty})} \lesssim \mathbb{I}^k_{\text{init}}[\Phi] t^{\alpha-\overline{\alpha}[\varphi]-2q}. \tag{8.139b}
$$

(2) There are pointwise-in-time estimates in the interior region, for $(t,r,\omega) \in \Omega^{\text{near}}_{t_0,\infty}$,

$$
|\varphi(t,r,\omega)|^2_{k-K,\mathbb{D}} \lesssim r^{-\underline{\alpha}[\varphi]} t^{-(\overline{\alpha}[\varphi]+1)+\underline{\alpha}[\varphi]} \mathbb{I}^k_{\text{init}}[\Phi]. \tag{8.140}
$$

Proof. For ease of presentation, we use mass normalization as in definition 4.4. Furthermore, the regularity constant K can vary from term to term, not merely, line to line. We put all the equations of the system (3.13) into the form of (8.132), and denote the corresponding right-hand side of each equation of $\varphi \in \{\hat{\sigma}', \widehat{G}_2, \hat{\tau}', \widehat{G}_1, \hat{\beta}', \widehat{G}_0\}$ by $\varrho[\varphi]$. The general strategy is to use estimate (6.42b) for $\mathcal{L}_\xi^j\hat{\psi}_{-2}$, estimates (8.123) for the transition flux, and lemma 8.16 applied to each transport equation in the system (3.13) to iteratively conclude that estimates (8.139) and (8.140) are valid. Since the part of the interior region $\{r \leq t\}$ to the future of Σ_{init} and to the past of $\Sigma_{t_0+h(t_0)}$ is compact, we can without loss of generality state our estimates in terms of $\mathbb{I}^k_{\text{init}}[\Phi]$. We will now discuss the proof of the energy and Morawetz estimate (8.139) for each of the fields and comment on the proof of the pointwise estimate (8.140) at the end of the proof.

For $\hat{\psi}_{-2}$, define

$$
\underline{\alpha}[\hat{\psi}_{-2}] = 2+, \qquad \overline{\alpha}[\hat{\psi}_{-2}] = 2l[\hat{\psi}_{-2}] + 2-. \tag{8.141}
$$

Observe that $\underline{\alpha}[\hat{\psi}_{-2}]$ does not conform to the formula for $\underline{\alpha}[\varphi]$ given in equation (8.138). For ease of reference, for $\varphi \in \Phi$, the values of $\underline{\alpha}[\varphi]$ and $\overline{\alpha}[\varphi]$ are given in the following table:

$$\begin{array}{c|cc} \varphi & \underline{\alpha}[\varphi] & \overline{\alpha}[\varphi] \\ \hline \hat{\psi}_{-2} & 2+ & 8- \\ \hat{\sigma}' & 0+ & 6- \\ \widehat{G}_2 & 0+ & 4- \\ \hat{\tau}' & 2+ & 8- \\ \widehat{G}_1 & 0+ & 2- \\ \hat{\beta}' & 4+ & 6- \\ \widehat{G}_0 & 2+ & 4- \end{array} .$$

In applying lemma 8.16, we shall freely make use of the fact that since $r = t$ on the transition surface Ξ, (8.123) can be restated in the form with explicit time decay, that is as estimate (8.133a) with the range of weights $\underline{\alpha}[\varphi] \le \alpha \le \overline{\alpha}[\varphi]$, for $\varphi \in \Phi\backslash\{\hat{\psi}_{-2}\}$.

The case $\hat{\psi}_{-2}$. From (6.42b), we get after a straightforward change of parameters, using $6- = \overline{\alpha}[\hat{\psi}_{-2}] - 2$, for $\underline{\alpha}[\hat{\psi}_{-2}] \le \alpha \le \overline{\alpha}[\hat{\psi}_{-2}]$,

$$\|\mathcal{L}_\xi^q \hat{\psi}_{-2}\|^2_{W^{k-K-q}_{\alpha-1}(\Omega^{\text{int}}_{t,\infty})} \lesssim t^{\alpha - \overline{\alpha}[\hat{\psi}_{-2}] - 2q} \mathbb{I}^k_{\text{init}}[\Phi]. \tag{8.142}$$

which is (8.139a).

The case $\hat{\sigma}'$. From estimate (8.123), we get hypothesis (8.133a) for $\underline{\alpha}[\hat{\sigma}'] \le \alpha \le \overline{\alpha}[\hat{\sigma}']$. From (8.142), for $\varrho[\hat{\sigma}'] = f\hat{\psi}_{-2}$ with $f = O_\infty(1)$, we get, after a reparametrization, hypothesis (8.133b) for the same range of weights. An application of lemma 8.16 proves point 1 for $\hat{\sigma}'$.

The case $\widehat{G}_2$. The argument for the $\widehat{G}_2$ follows exactly the same pattern, which establishes point 1 for $\widehat{G}_2$.

The case $\hat{\tau}'$. From the transport equation (3.13c), we have

$$\|\mathcal{L}_\xi^q \varrho[\hat{\tau}']\|^2_{W^{k-K-q}_{\alpha+1}(\Omega^{\text{int}}_{t,\infty})} \lesssim \|\mathcal{L}_\xi^q \hat{\sigma}'\|^2_{W^{k-K-q}_{\alpha-3}(\Omega^{\text{int}}_{t,\infty})}. \tag{8.143}$$

Making the substitution $\alpha - 3 = \alpha + 1 - 2m[\hat{\tau}'] = \beta - 1$, or $\beta = \alpha - 2(m[\hat{\tau}'] - 1)$, we find using estimate (8.139b) for $\hat{\sigma}'$, after a reparametrization, that $\varrho[\hat{\tau}']$ satisfies hypothesis (8.133b) for the range of weights $\underline{\alpha}[\hat{\tau}'] \le \alpha \le \overline{\alpha}[\hat{\tau}']$, where

$$\underline{\alpha}[\hat{\tau}'] = \underline{\alpha}[\hat{\sigma}'] + 2(m[\hat{\tau}'] - 1) = 2+, \tag{8.144a}$$

$$\overline{\alpha}[\hat{\tau}'] = \overline{\alpha}[\hat{\sigma}'] + 2(m[\hat{\tau}'] - 1) = 8- . \tag{8.144b}$$

On the other hand, we have that estimate (8.133a) holds for the range $0+ \le \alpha \le \overline{\alpha}[\hat{\tau}']$. We may thus apply lemma 8.16 for the intersection of these ranges, $\underline{\alpha}[\hat{\tau}'] \le \alpha \le \overline{\alpha}[\hat{\tau}']$ to prove point 1 for $\hat{\tau}'$.

The case $\widehat{G}_1$. We have that

$$\|\mathcal{L}_\xi^q \varrho[\widehat{G}_1]\|^2_{W^{k-K-q}_{\alpha+1}(\Omega^{\text{int}}_{t,\infty})} \lesssim \|\mathcal{L}_\xi^q \hat{\tau}'\|^2_{W^{k-K-q}_{\alpha+5}(\Omega^{\text{int}}_{t,\infty})} + \|\mathcal{L}_\xi^q \widehat{G}_2\|^2_{W^{k-K-q}_{\alpha+1}(\Omega^{\text{int}}_{t,\infty})}. \tag{8.145}$$

We consider the second term on the right-hand side first. Writing $\alpha+1 = \beta-1$ and using estimate (8.139b) for $\widehat{G}_2$, we have, after a reparametrization,

$$\|\mathcal{L}_\xi^q \widehat{G}_2\|^2_{W^{k-K-q}_{\alpha+1}(\Omega^{\text{int}}_{t,\infty})} \lesssim t^{\alpha - (\overline{\alpha}[\widehat{G}_2] - 2) - 2q} \mathbb{I}^k_{\text{init}}[\Phi] \tag{8.146}$$

for $\underline{\alpha}[\widehat{G}_2] - 2 \le \alpha \le \overline{\alpha}[\widehat{G}_2] - 2$. Here we must restrict the lower limit to zero, which yields the range $0+ \le \alpha \le 2-$. For the first term, from the estimates for $\hat{\tau}'$, we get after the substitution $\alpha \mapsto \alpha + 6$,

$$\|\mathcal{L}_\xi^q \hat{\tau}'\|^2_{W^{k-K-q}_{\alpha+5}(\Omega^{\text{int}}_{t,\infty})} \lesssim t^{\alpha - (\overline{\alpha}[\hat{\tau}'] - 6) - 2q} \mathbb{I}^k_{\text{init}}[\Phi] \tag{8.147}$$

for the range $\underline{\alpha}[\hat{\tau}'] - 6 \le \alpha \le \overline{\alpha}[\hat{\tau}'] - 6$, which is $-4+ \le \alpha \le 2-$, which is less restrictive than the one arising from $\widehat{G}_2$. Thus, we find that estimate (8.133b) holds for $\varrho[\widehat{G}_1]$ for the range of weights $\underline{\alpha}[\widehat{G}_1] \le \alpha \le \overline{\alpha}[\widehat{G}_1]$ with $\underline{\alpha}[\widehat{G}_1] = 0+$, $\overline{\alpha}[\widehat{G}_1] = 2-$. This proves point 1 for $\widehat{G}_1$.

The case $\hat\beta'$. We have

$$\|\varrho[\hat\beta']\|^2_{W^{k-K-q}_{\alpha+1}(\Omega^{\text{int}}_{t,\infty})} \lesssim \|\mathcal{L}^q_\xi \widehat{G}_1\|^2_{W^{k-K-q}_{\alpha-5}(\Omega^{\text{int}}_{t,\infty})} + \|\mathcal{L}^q_\xi \widehat{G}_2\|^2_{W^{k-K-q}_{\alpha-5}(\Omega^{\text{int}}_{t,\infty})}. \tag{8.148}$$

Making the substitution $\alpha-5=\beta-1$, we get estimates for the ranges $\underline{\alpha}[\widehat{G}_1]+4<\alpha<\overline{\alpha}[\widehat{G}_1]+4$, and $\underline{\alpha}[\widehat{G}_2]+4 \le \alpha \le \overline{\alpha}[\widehat{G}_2]+4$, respectively. Here the case $\widehat{G}_1$ gives the more restrictive range, and we find that estimate (8.133b) holds for $\varrho[\hat\beta']$ for the range $\underline{\alpha}[\hat\beta'] \le \alpha \le \overline{\alpha}[\hat\beta']$ with $\underline{\alpha}[\hat\beta'] = 2(m[\hat\beta']-1)+ = 4+$, $\overline{\alpha}[\hat\beta'] = 2l[\hat\beta']+2- = 6-$. This proves point 1 for $\hat\beta'$.

The case $\widehat{G}_0$. We have

$$\|\varrho[\widehat{G}_0]\|^2_{W^{k-K-q-1}_{\alpha+1}(\Omega^{\text{int}}_{t,\infty})} \lesssim \|\mathcal{L}^q_\xi \widehat{G}_1\|^2_{W^{k-K-q}_{\alpha-3}(\Omega^{\text{int}}_{t,\infty})} + \|\mathcal{L}^q_\xi \hat\beta'\|^2_{W^{k-K-q}_{\alpha+1}(\Omega^{\text{int}}_{t,\infty})}. \tag{8.149}$$

Proceeding as above yields for the first term

$$\|\mathcal{L}^q_\xi \widehat{G}_1\|^2_{W^{k-K-q}_{\alpha-3}(\Omega^{\text{int}}_{t,\infty})} \lesssim t^{\alpha-(\overline{\alpha}[\widehat{G}_1]+2)-2q}\mathbb{I}^k_{\text{init}}[\Phi] \tag{8.150}$$

for the range $\underline{\alpha}[\widehat{G}_1]+2 \le \alpha \le \overline{\alpha}[\widehat{G}_1]+2$, i.e. $2+ \le \alpha \le 4-$. Analogously, for the second term we get for the range $\underline{\alpha}[\hat\beta']-2 \le \alpha \le \overline{\alpha}[\hat\beta']-2$, i.e. $2+ \le \alpha \le 4-$,

$$\|\mathcal{L}^q_\xi \hat\beta'\|^2_{W^{k-K-q}_{\alpha+1}(\Omega^{\text{int}}_{t,\infty})} \lesssim t^{\alpha-(\overline{\alpha}[\hat\beta']-2)-2q}\mathbb{I}^k_{\text{init}}[\Phi] \tag{8.151}$$

This proves (8.133b) for the range $\underline{\alpha}[\widehat{G}_0] \le \alpha \le \overline{\alpha}[\widehat{G}_0]$, with $\underline{\alpha}[\widehat{G}_0] = 2(m[\widehat{G}_0]-1)+ = 2+$, $\overline{\alpha}[\widehat{G}_0] = 2l[\widehat{G}_0]+2- = 4-$, and hence completes the proof of point 1 for $\varphi \in \Phi\backslash\{\hat\psi_{-2}\}$.

It remains to consider point 2. For $\varphi \in \Phi\backslash\{\hat\psi_{-2}\}$, this follows from estimate (8.134b) with $\alpha = \underline{\alpha}[\varphi]$. □

8.6. **Proof of the main theorems 1.1 and 1.5.** This section completes the proofs of the theorems from the introduction.

Proof of theorem 1.5. If δg satisfies the linearized Einstein equation in the outgoing radiation gauge and satisfies the basic decay condition of definition 1.3, then it corresponds to an outgoing BEAM solution of the linearized Einstein equation as in definition 8.2. Thus, lemmas 8.15 and 8.18 can be applied. These yield that, for $\varphi \in \{\widehat{G}_i\}_{i=0}^2$, and $k \in \mathbb{N}$ sufficiently large,

$$|\varphi|^2 \lesssim \begin{cases} r^{-2m[\varphi]}t^{2m[\varphi]-3-2l[\varphi]+}\mathbb{I}^{k-2}_{\text{init}}[\Phi] & \text{if } r \ge t, \\ r^{-2\max\{m[\varphi]-1,0\}-}t^{-(2l[\varphi]+3)+2\max\{m[\varphi]-1,0\}+}\mathbb{I}^{k-2}_{\text{init}}[\Phi] & \text{if } r \le t. \end{cases} \tag{8.152}$$

Equation (3.12) relates the $\widehat{G}_i$ to the $G_{i0'}$ by a rescaling by some rational factor that grows as a particular power in r, which will be denoted by $p[\varphi]$ in this paragraph. From definition 8.13 and equation (3.12), the relevant parameters are given in the following table:

φ	m	ℓ	p
$\widehat{G}_2$	0	1	1
$\widehat{G}_1$	0	0	2
$\widehat{G}_0$	2	1	1

Thus, one finds, in the exterior region,

$$|G_{20'}|^2 \lesssim r^{-2}t^{-5+}\mathbb{I}^{k-2}_{\text{init}}[\Phi], \tag{8.153a}$$
$$|G_{10'}|^2 \lesssim r^{-4}t^{-3+}\mathbb{I}^{k-2}_{\text{init}}[\Phi], \tag{8.153b}$$
$$|G_{00'}|^2 \lesssim r^{-6}t^{-1+}\mathbb{I}^{k-2}_{\text{init}}[\Phi], \tag{8.153c}$$

and, in the interior region,

$$|G_{20'}|^2 \lesssim r^{-2-}t^{-5+}\mathbb{I}^{k-2}_{\text{init}}[\Phi], \tag{8.154a}$$
$$|G_{10'}|^2 \lesssim r^{-4-}t^{-3+}\mathbb{I}^{k-2}_{\text{init}}[\Phi], \tag{8.154b}$$
$$|G_{00'}|^2 \lesssim r^{-4-}t^{-3+}\mathbb{I}^{k-2}_{\text{init}}[\Phi]. \tag{8.154c}$$

Recall that the fields $\varphi \in \Phi$ are defined in definition 8.14 in terms of the linearized metric δg_{ab} and its derivatives up to second order as specified in section 3.1. From these definitions, the definition of the initial data norm $\mathbb{I}^{k-2}_{\text{init}}[\Phi]$ in definition 8.14, and the definition of $\|\delta g\|^2_{H^k_7(\Sigma_{\text{init}})}$ in equation (1.10), it is straightforward to verify that

$$\mathbb{I}^{k-2}_{\text{init}}[\Phi] \lesssim \|\delta g\|^2_{H^k_7(\Sigma_{\text{init}})}. \tag{8.155}$$

This completes the proof of theorem 1.5. □

Proof of theorem 1.1. From [32], it is known that, for $|a|/M$ sufficiently small and $k \in \mathbb{N}$ sufficiently large, the BEAM condition for $\hat{\psi}_{-2}$ from definition 6.8 holds, and, also, the BEAM condition 2 for $\hat{\psi}_{+2}$ from definition 7.1 holds. Moreover, there is a bound $\mathbb{I}^{k-2;1}_{\text{init}}(\hat{\psi}_{+2}) \lesssim \|\delta g\|^2_{H^k_7(\Sigma_{\text{init}})}$, which is finite by assumption. Thus, theorem 7.8 implies, for $|a|/M$ sufficiently small, the pointwise condition 3 for $\hat{\psi}_{+2}$ from definition 7.1 holds. The BEAM condition from definition 6.8 for $\hat{\psi}_{-2}$ and the pointwise condition 3 from definition 7.1 for $\hat{\psi}_{+2}$ imply the basic decay conditions of definition 1.3. Since the basic decay conditions of definition 1.3 holds for $|a|/M$ sufficiently small, theorem 1.5 immediately implies the desired estimates, which completes the proof. □

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Appendix A. Field equations

A.1. **Linearized Einstein vacuum equations.** In this appendix, we give the component form in GHP notation of the linearized Einstein field equations which are used in this paper. The structure equations (3.2) in general take the form

$$\tilde{\beta} = \tfrac{1}{4}(\text{þ} + 2\rho - \bar{\rho})G_{12'} - \tfrac{1}{4}(\text{þ}' + \rho' - 2\bar{\rho}')G_{01'} - \tfrac{1}{4}(\eth + 2\tau - 2\bar{\tau}')G_{11'} + \tfrac{1}{4}(\eth' - \bar{\tau} + \tau')G_{02'} + \tfrac{1}{16}\eth \not{G}, \tag{A.1a}$$

$$\tilde{\beta}' = -\tfrac{1}{4}(\text{þ} + \rho - 2\bar{\rho})G_{21'} + \tfrac{1}{4}(\text{þ}' + 2\rho' - \bar{\rho}')G_{10'} + \tfrac{1}{4}(\eth + \tau - \bar{\tau}')G_{20'} - \tfrac{1}{4}(\eth' - 2\bar{\tau} + 2\tau')G_{11'} + \tfrac{1}{16}\eth' \not{G}, \tag{A.1b}$$

$$\tilde{\epsilon} = \tfrac{1}{4}(\text{þ} + 2\rho - 2\bar{\rho})G_{11'} - \tfrac{1}{4}(\text{þ}' + \rho' - \bar{\rho}')G_{00'} - \tfrac{1}{4}(\eth + 2\tau - \bar{\tau}')G_{10'} + \tfrac{1}{4}(\eth' - 2\bar{\tau} + \tau')G_{01'} + \tfrac{1}{16}\text{þ} \not{G}, \tag{A.1c}$$

$$\tilde{\epsilon}' = -\tfrac{1}{4}(\text{þ} + \rho - \bar{\rho})G_{22'} + \tfrac{1}{4}(\text{þ}' + 2\rho' - 2\bar{\rho}')G_{11'} + \tfrac{1}{4}(\eth + \tau - 2\bar{\tau}')G_{21'} - \tfrac{1}{4}(\eth' - \bar{\tau} + 2\tau')G_{12'} + \tfrac{1}{16}\text{þ}' \not{G}, \tag{A.1d}$$

$$\tilde{\kappa} = \tfrac{1}{2}(\text{þ} - 2\bar{\rho})G_{01'} - \tfrac{1}{2}(\eth - \bar{\tau}')G_{00'}, \tag{A.1e}$$

$$\tilde{\kappa}' = \tfrac{1}{2}(\text{þ}' - 2\bar{\rho}')G_{21'} - \tfrac{1}{2}(\eth' - \bar{\tau})G_{22'}, \tag{A.1f}$$

$$\tilde{\rho} = -\tfrac{1}{2}G_{00'}\rho' + \tfrac{1}{2}G_{01'}\tau' + \tfrac{1}{2}(\text{þ} - 2\bar{\rho})G_{11'} - \tfrac{1}{2}(\eth - \bar{\tau}')G_{10'} - \tfrac{1}{8}\text{þ} \not{G}, \tag{A.1g}$$

$$\tilde{\rho}' = -\tfrac{1}{2}\rho G_{22'} + \tfrac{1}{2}\tau G_{21'} + \tfrac{1}{2}(\text{þ}' - 2\bar{\rho}')G_{11'} - \tfrac{1}{2}(\eth' - \bar{\tau})G_{12'} - \tfrac{1}{8}\text{þ}' \not{G}, \tag{A.1h}$$

$$\tilde{\sigma} = \tfrac{1}{2}(\text{þ} - \bar{\rho})G_{02'} - \tfrac{1}{2}(\eth - 2\bar{\tau}')G_{01'}, \tag{A.1i}$$

$$\tilde{\sigma}' = \tfrac{1}{2}(\text{þ}' - \bar{\rho}')G_{20'} - \tfrac{1}{2}(\eth' - 2\bar{\tau})G_{21'}, \tag{A.1j}$$

$$\tilde{\tau} = -\tfrac{1}{2}\rho' G_{01'} + \tfrac{1}{2}\tau' G_{02'} + \tfrac{1}{2}(\text{þ} - \bar{\rho})G_{12'} - \tfrac{1}{2}(\eth - 2\bar{\tau}')G_{11'} - \tfrac{1}{8}\eth \not{G}, \tag{A.1k}$$

$$\tilde{\tau}' = -\tfrac{1}{2}\rho G_{21'} + \tfrac{1}{2}\tau G_{20'} + \tfrac{1}{2}(\text{þ}' - \bar{\rho}')G_{10'} - \tfrac{1}{2}(\eth' - 2\bar{\tau})G_{11'} - \tfrac{1}{8}\eth' \not{G}. \tag{A.1l}$$

The linearized vacuum Einstein equations (3.3b) and (3.3c) are

$$\begin{aligned}0 = &-(\text{þ}-\rho-\bar\rho)\tilde\epsilon' + (\text{þ}-\rho-\bar\rho)\tilde\rho' - (\text{þ}'-\rho'-\bar\rho')\tilde\epsilon + (\text{þ}'-\rho'-\bar\rho')\tilde\rho + (\eth-\tau-\bar\tau')\tilde\beta' \\ &-(\eth-\tau-\bar\tau')\tilde\tau' + (\eth'-\bar\tau-\tau')\tilde\beta - (\eth'-\bar\tau-\tau')\tilde\tau,\end{aligned} \tag{A.2a}$$

$$0 = (\text{þ}'-\rho')\tilde\sigma - (\eth-\tau)\tilde\tau - \tfrac12 G_{02'}\Psi_2 + 2\tau\tilde\beta, \tag{A.2b}$$

$$0 = (\text{þ}'-\rho')\tilde\beta + (\eth-\tau)\tilde\epsilon' + G_{12'}\Psi_2 - \tau\tilde\rho' - \rho'\tilde\tau, \tag{A.2c}$$

$$0 = -(\text{þ}'-\rho')\tilde\rho' + (\eth-\tau)\tilde\kappa' - \tfrac12\Psi_2 G_{22'} + 2\tilde\epsilon'\rho', \tag{A.2d}$$

$$0 = -(\text{þ}-\rho)\tilde\rho + (\eth'-\tau')\tilde\kappa - \tfrac12\Psi_2 G_{00'} + 2\tilde\epsilon\rho, \tag{A.2e}$$

$$0 = -\tfrac12(\text{þ}-\rho+\bar\rho)\tilde\tau + \tfrac12(\text{þ}'-\rho'+\bar\rho')\tilde\kappa - \tfrac12(\eth-\tau+\bar\tau')\tilde\rho + \tfrac12(\eth'+\bar\tau-\tau')\tilde\sigma - \tfrac12\Psi_2 G_{01'} + \tilde\beta\rho + \tilde\epsilon\tau, \tag{A.2f}$$

$$0 = (\text{þ}-\rho)\tilde\beta' + (\eth'-\tau')\tilde\epsilon + \Psi_2 G_{10'} - \tau'\tilde\rho - \rho\tilde\tau', \tag{A.2g}$$

$$\begin{aligned}0 = &\tfrac12(\text{þ}-\rho+\bar\rho)\tilde\epsilon' + \tfrac12(\text{þ}'-\rho'+\bar\rho')\tilde\epsilon + \tfrac12(\eth-\tau+\bar\tau')\tilde\beta' + \tfrac12(\eth'+\bar\tau-\tau')\tilde\beta + \Psi_2 G_{11'} - \tfrac12\rho\tilde\rho' - \tfrac12\rho'\tilde\rho \\ &-\tfrac12\tau\tilde\tau' - \tfrac12\tau'\tilde\tau,\end{aligned} \tag{A.2h}$$

$$0 = (\text{þ}-\rho)\tilde\sigma' - (\eth'-\tau')\tilde\tau' - \tfrac12\Psi_2 G_{20'} + 2\tau'\tilde\beta', \tag{A.2i}$$

$$0 = \tfrac12(\text{þ}-\rho+\bar\rho)\tilde\kappa' - \tfrac12(\text{þ}'-\rho'+\bar\rho')\tilde\tau' + \tfrac12(\eth-\tau+\bar\tau')\tilde\sigma' - \tfrac12(\eth'+\bar\tau-\tau')\tilde\rho' - \tfrac12\Psi_2 G_{21'} + \rho'\tilde\beta' + \tau'\tilde\epsilon'. \tag{A.2j}$$

The remaining Ricci relations (3.3d) are

$$\vartheta\Psi_0 = (\text{þ}-\bar\rho)\tilde\sigma - (\eth-\bar\tau')\tilde\kappa, \tag{A.3a}$$

$$\begin{aligned}\vartheta\Psi_1 = &\tfrac12(\text{þ}+\rho-\bar\rho)\tilde\beta + \tfrac14(\text{þ}+\rho-\bar\rho)\tilde\tau - \tfrac14(\text{þ}'+3\rho'-\bar\rho')\tilde\kappa - \tfrac12(\eth+\tau-\bar\tau')\tilde\epsilon - \tfrac14(\eth+\tau-\bar\tau')\tilde\rho \\ &+\tfrac14(\eth'-\bar\tau+3\tau')\tilde\sigma,\end{aligned} \tag{A.3b}$$

$$\begin{aligned}\vartheta\Psi_2 = &-\tfrac14\Psi_2\not{G} - \tfrac13(\text{þ}+2\rho-\bar\rho)\tilde\epsilon' - \tfrac16(\text{þ}+2\rho-\bar\rho)\tilde\rho' - \tfrac13(\text{þ}'+2\rho'-\bar\rho')\tilde\epsilon - \tfrac16(\text{þ}'+2\rho'-\bar\rho')\tilde\rho \\ &+\tfrac13(\eth+2\tau-\bar\tau')\tilde\beta' + \tfrac16(\eth+2\tau-\bar\tau')\tilde\tau' + \tfrac13(\eth'-\bar\tau+2\tau')\tilde\beta + \tfrac16(\eth'-\bar\tau+2\tau')\tilde\tau,\end{aligned} \tag{A.3c}$$

$$\begin{aligned}\vartheta\Psi_3 = &-\tfrac14(\text{þ}+3\rho-\bar\rho)\tilde\kappa' + \tfrac12(\text{þ}'+\rho'-\bar\rho')\tilde\beta' + \tfrac14(\text{þ}'+\rho'-\bar\rho')\tilde\tau' + \tfrac14(\eth+3\tau-\bar\tau')\tilde\sigma' \\ &-\tfrac12(\eth'-\bar\tau+\tau')\tilde\epsilon' - \tfrac14(\eth'-\bar\tau+\tau')\tilde\rho',\end{aligned} \tag{A.3d}$$

$$\vartheta\Psi_4 = (\text{þ}'-\bar\rho')\tilde\sigma' - (\eth'-\bar\tau)\tilde\kappa'. \tag{A.3e}$$

We also have the commutator relations (3.3a)

$$0 = 2(\text{þ}-\rho-\bar\rho)\tilde\beta - (\text{þ}+\rho-\bar\rho)\tilde\tau + (\text{þ}'-\rho'-\bar\rho')\tilde\kappa - 2(\eth-\tau-\bar\tau')\tilde\epsilon + (\eth+\tau-\bar\tau')\tilde\rho - (\eth'-\bar\tau-\tau')\tilde\sigma, \tag{A.4a}$$

$$0 = -(\text{þ}-\bar\rho)\tilde\rho' + (\text{þ}'-\bar\rho')\tilde\rho + (\eth-\bar\tau')\tilde\tau' - (\eth'-\bar\tau)\tilde\tau + 2\rho\tilde\epsilon' - 2\rho'\tilde\epsilon - 2\tau\tilde\beta' + 2\tau'\tilde\beta, \tag{A.4b}$$

$$\begin{aligned}0 = &-(\text{þ}-\rho-\bar\rho)\tilde\kappa' - 2(\text{þ}'-\rho'-\bar\rho')\tilde\beta' + (\text{þ}'+\rho'-\bar\rho')\tilde\tau' + (\eth-\tau-\bar\tau')\tilde\sigma' + 2(\eth'-\bar\tau-\tau')\tilde\epsilon' \\ &-(\eth'-\bar\tau+\tau')\tilde\rho',\end{aligned} \tag{A.4c}$$

and reality conditions $\bar{\not{G}}_{A'A} = \not{G}_{AA'}$

$$\bar{\tilde\epsilon} - \bar{\tilde\rho} = \tilde\epsilon - \tilde\rho, \qquad \bar{\tilde\beta} - \bar{\tilde\tau} = \tilde\beta' - \tilde\tau', \qquad \bar{\tilde\beta'} - \bar{\tilde\tau'} = \tilde\beta - \tilde\tau, \qquad \bar{\tilde\epsilon'} - \bar{\tilde\rho'} = \tilde\epsilon' - \tilde\rho'. \tag{A.5}$$

Furthermore, the linearized vacuum Bianchi equations (3.3e) take the form

$$0 = (\text{þ}'-\rho')\vartheta\Psi_0 - (\eth-4\tau)\vartheta\Psi_1 - \tfrac32\Psi_2\rho G_{02'} - 3\Psi_2\tilde\sigma + \tfrac32\Psi_2\tau G_{01'}, \tag{A.6a}$$

$$0 = (\text{þ}'-2\rho')\vartheta\Psi_1 - (\eth-3\tau)\vartheta\Psi_2 + 3G_{12'}\Psi_2\rho + \tfrac32\Psi_2\rho' G_{01'} - 3G_{11'}\Psi_2\tau - \tfrac32\Psi_2\tau' G_{02'} + 3\Psi_2\tilde\tau, \tag{A.6b}$$

$$0 = (\text{þ}'-3\rho')\vartheta\Psi_2 - (\eth-2\tau)\vartheta\Psi_3 - \tfrac32\Psi_2\rho G_{22'} - 3\Psi_2\rho' G_{11'} - 3\Psi_2\tilde\rho' + \tfrac32\Psi_2\tau G_{21'} + 3\Psi_2\tau' G_{12'}, \tag{A.6c}$$

$$0 = (\text{þ}'-4\rho')\vartheta\Psi_3 - (\eth-\tau)\vartheta\Psi_4 + 3\Psi_2\tilde\kappa' + \tfrac32\Psi_2\rho' G_{21'} - \tfrac32\Psi_2\tau' G_{22'}, \tag{A.6d}$$

$$0 = (\text{þ}-4\rho)\vartheta\Psi_1 - (\eth'-\tau')\vartheta\Psi_0 + 3\Psi_2\tilde\kappa + \tfrac32\Psi_2\rho G_{01'} - \tfrac32\Psi_2\tau G_{00'}, \tag{A.6e}$$

$$0 = (\text{þ} - 3\rho)\vartheta\Psi_2 - (\eth' - 2\tau')\vartheta\Psi_1 - 3\Psi_2\rho G_{11'} - \tfrac{3}{2}\Psi_2\rho' G_{00'} - 3\Psi_2\tilde{\rho} + 3\Psi_2\tau G_{10'} + \tfrac{3}{2}\Psi_2\tau' G_{01'}, \tag{A.6f}$$

$$0 = (\text{þ} - 2\rho)\vartheta\Psi_3 - (\eth' - 3\tau')\vartheta\Psi_2 + \tfrac{3}{2}\Psi_2\rho G_{21'} + 3\Psi_2\rho' G_{10'} - \tfrac{3}{2}\Psi_2\tau G_{20'} - 3\Psi_2\tau' G_{11'} + 3\Psi_2\tilde{\tau}', \tag{A.6g}$$

$$0 = (\text{þ} - \rho)\vartheta\Psi_4 - (\eth' - 4\tau')\vartheta\Psi_3 - \tfrac{3}{2}\Psi_2\rho' G_{20'} - 3\Psi_2\tilde{\sigma}' + \tfrac{3}{2}\Psi_2\tau' G_{21'}. \tag{A.6h}$$

A.2. **Linearized Einstein field equations in ORG.** A calculation using the relations (3.8) and (3.9) yield the following lemma, which we state for completeness. Observe however that the proof of lemma 3.8 is directly referring to the equations in appendix A.1.

Lemma A.1. *Under the ORG condition the vacuum linearized Einstein equations can be organized as the transport equations*

$$\text{þ}' G_{00'} = -4\tilde{\epsilon} + 2\tilde{\rho} - 2\overline{\tilde{\rho}} - 2G_{10'}\tau - 2G_{01'}\bar{\tau} + G_{01'}\tau' + G_{10'}\bar{\tau}', \tag{A.7a}$$

$$(\text{þ}' - \rho')G_{01'} = -G_{02'}\bar{\tau} + 2\overline{\tilde{\tau}'}, \tag{A.7b}$$

$$(\text{þ}' - \rho')G_{02'} = 2\overline{\tilde{\sigma}'}, \tag{A.7c}$$

$$(\text{þ}' - \bar{\rho}')G_{10'} = -G_{20'}\tau + 2\tilde{\tau}', \tag{A.7d}$$

$$(\text{þ}' - \bar{\rho}')G_{20'} = 2\tilde{\sigma}', \tag{A.7e}$$

$$(\text{þ}' - \rho' + \bar{\rho}')\tilde{\tau}' = 2\tilde{\beta}'\rho' + (\eth - \tau + \bar{\tau}')\tilde{\sigma}', \tag{A.7f}$$

$$(\text{þ}' - 2\rho' - \bar{\rho}')\tilde{\beta}' = \rho'\tilde{\tau}' - \bar{\rho}'\tilde{\tau}' + (\eth - \tau)\tilde{\sigma}', \tag{A.7g}$$

$$(\text{þ}' - \bar{\rho}')\tilde{\sigma}' = \vartheta\Psi_4, \tag{A.7h}$$

$$\begin{aligned}(\text{þ}' - \rho' - \bar{\rho}')\tilde{\rho} = {}& \tilde{\epsilon}\rho' + \overline{\tilde{\epsilon}}\rho' + \tfrac{1}{2}G_{00'}\rho'\bar{\rho}' + 2\tilde{\beta}'\tau + \tfrac{1}{2}G_{10'}\bar{\rho}'\tau - G_{01'}\bar{\rho}'\tau' - \tfrac{1}{2}G_{02'}\bar{\tau}\tau' + \tau'\overline{\tilde{\tau}'} \\ &- (\eth - \bar{\tau}')\tilde{\tau}',\end{aligned} \tag{A.7i}$$

$$\begin{aligned}(\text{þ}' + \bar{\rho}')\tilde{\kappa} = {}& \tfrac{5}{4}G_{01'}\Psi_2 + \frac{G_{01'}\bar{\Psi}_2\overline{\kappa}_{1'}}{4\kappa_1} - 2\tilde{\beta}\rho - \tfrac{3}{2}G_{01'}\bar{\rho}\rho' - 2\tilde{\epsilon}\tau - \tfrac{1}{2}G_{00'}\bar{\rho}'\tau + \tfrac{1}{2}G_{02'}\rho\tau' \\ &+ G_{02'}\bar{\rho}\tau' + \tfrac{1}{2}\tau'(\eth - \tau - \bar{\tau}')G_{01'} + (\eth - \tau + \bar{\tau}')\tilde{\rho} - (\eth' + \bar{\tau} - 2\tau')\tilde{\sigma} - \tfrac{1}{2}\rho'\,\eth\, G_{00'},\end{aligned} \tag{A.7j}$$

$$(\text{þ}' - 2\rho')\tilde{\epsilon} = \tilde{\beta}'\tau - \tilde{\beta}\bar{\tau} - \tilde{\beta}\tau' - \tfrac{1}{2}G_{01'}\rho'\tau' + \tfrac{1}{2}G_{02'}\tau'^2 - \tilde{\beta}'\bar{\tau}' - (\eth - \tau - \bar{\tau}')\tilde{\tau}', \tag{A.7k}$$

$$(\text{þ}' - \rho')\tilde{\beta} = -\tfrac{1}{2}G_{01'}\rho'^2 + \tfrac{1}{2}G_{02'}\rho'\tau', \tag{A.7l}$$

$$\begin{aligned}(\text{þ}' - \rho')\tilde{\sigma} = {}& \tfrac{3}{4}G_{02'}\Psi_2 - \frac{G_{02'}\bar{\Psi}_2\overline{\kappa}_{1'}}{4\kappa_1} + \tfrac{1}{2}G_{02'}\rho\rho' - \tfrac{1}{2}G_{02'}\bar{\rho}\rho' - 2\tilde{\beta}\tau - \tfrac{1}{2}\rho'(\eth + \tau)G_{01'} \\ &+ \tfrac{1}{2}\tau'\,\eth\, G_{02'},\end{aligned} \tag{A.7m}$$

$$(\text{þ}' - 4\rho')\vartheta\Psi_3 = (\eth - \tau)\vartheta\Psi_4, \tag{A.7n}$$

$$(\text{þ}' - 3\rho')\vartheta\Psi_2 = (\eth - 2\tau)\vartheta\Psi_3, \tag{A.7o}$$

$$(\text{þ}' - 2\rho')\vartheta\Psi_1 = (\eth - 3\tau)\vartheta\Psi_2, \tag{A.7p}$$

$$(\text{þ}' - \rho')\vartheta\Psi_0 = \tfrac{3}{2}G_{02'}\Psi_2\rho + 3\Psi_2\tilde{\sigma} - \tfrac{3}{2}G_{01'}\Psi_2\tau + (\eth - 4\tau)\vartheta\Psi_1, \tag{A.7q}$$

together with the set

$$\tilde{\beta} = -\tfrac{1}{2}G_{01'}\rho' + \tfrac{1}{2}G_{01'}\bar{\rho}' - \tfrac{1}{2}\overline{\tilde{\tau}'} + \tfrac{1}{4}(\eth' + \tau')G_{02'}, \tag{A.8a}$$

$$\tilde{\beta}' = \tfrac{1}{2}G_{10'}\rho' + \tfrac{1}{2}\tilde{\tau}' + \tfrac{1}{4}(\eth - \bar{\tau}')G_{20'}, \tag{A.8b}$$

$$\tilde{\kappa} = \tfrac{1}{2}(\text{þ} - 2\bar{\rho})G_{01'} - \tfrac{1}{2}(\eth - \bar{\tau}')G_{00'}, \tag{A.8c}$$

$$\tilde{\rho} = -\tfrac{1}{2}G_{00'}\rho' + \tfrac{1}{2}G_{01'}\tau' - \tfrac{1}{2}(\eth - \bar{\tau}')G_{10'}, \tag{A.8d}$$

$$\tilde{\sigma} = \tfrac{1}{2}(\text{þ} - \bar{\rho})G_{02'} - \tfrac{1}{2}(\eth - 2\bar{\tau}')G_{01'}, \tag{A.8e}$$

$$\tilde{\tau} = -\tfrac{1}{2}G_{01'}\rho' + \tfrac{1}{2}G_{02'}\tau', \tag{A.8f}$$

$$(\text{þ} - \rho)\tilde{\rho} = -\tfrac{1}{2}G_{00'}\Psi_2 + 2\tilde{\epsilon}\rho + (\eth' - \tau')\tilde{\kappa}, \tag{A.8g}$$

$$(\text{þ} - 2\rho - \bar{\rho})\tilde{\beta} = -\tfrac{1}{2}G_{01'}(\Psi_2 + \rho\rho' - \bar{\rho}\rho') + \tilde{\kappa}\bar{\rho}' + \tfrac{1}{2}G_{02'}(\rho - \bar{\rho})\tau' + (\eth - \bar{\tau}')\tilde{\epsilon} + (\eth' - \tau')\tilde{\sigma} - \eth\,\tilde{\rho}, \tag{A.8h}$$

$$(\text{þ} - \rho)\tilde{\beta}' = -G_{10'}\Psi_2 + \tilde{\rho}\tau' + \rho\tilde{\tau}' - (\eth' - \tau')\tilde{\epsilon}, \tag{A.8i}$$

$$(\text{þ} - \rho)\tilde{\sigma}' = \tfrac{1}{2}G_{20'}\Psi_2 - 2\tilde{\beta}'\tau' + (\eth' - \tau')\tilde{\tau}', \tag{A.8j}$$

$$0 = \tilde{\epsilon}(\rho' + \bar{\rho}') - \rho'\tilde{\rho} - (\eth - \bar{\tau}')\tilde{\tau}' + (\eth' - 2\tau')\tilde{\beta} + \eth\,\tilde{\beta}', \tag{A.8k}$$

$$\vartheta\Psi_0 = (\text{þ} - \bar{\rho})\tilde{\sigma} - (\eth - \bar{\tau}')\tilde{\kappa}, \tag{A.8l}$$

$$\vartheta\Psi_1 = -\tilde{\kappa}\rho' + \tilde{\sigma}\tau' + (\text{þ} - \bar{\rho})\tilde{\beta} - (\eth - \bar{\tau}')\tilde{\epsilon}, \tag{A.8m}$$

$$\vartheta\Psi_2 = -2\tilde{\epsilon}\rho' + 2\tilde{\beta}\tau' + (\eth - \bar{\tau}')\tilde{\tau}', \tag{A.8n}$$

$$\vartheta\Psi_3 = 2\tilde{\beta}'\rho' + (\rho' - \bar{\rho}')\tilde{\tau}' + \eth\,\tilde{\sigma}', \tag{A.8o}$$

$$0 = 3\Psi_2\tilde{\kappa} + \tfrac{3}{2}G_{01'}\Psi_2\rho - \tfrac{3}{2}G_{00'}\Psi_2\tau + (\text{þ} - 4\rho)\vartheta\Psi_1 - (\eth' - \tau')\vartheta\Psi_0, \tag{A.8p}$$

$$0 = -\tfrac{3}{2}G_{00'}\Psi_2\rho' - 3\Psi_2\tilde{\rho} + 3G_{10'}\Psi_2\tau + \tfrac{3}{2}G_{01'}\Psi_2\tau' + (\text{þ} - 3\rho)\vartheta\Psi_2 - (\eth' - 2\tau')\vartheta\Psi_1, \tag{A.8q}$$

$$0 = 3G_{10'}\Psi_2\rho' - \tfrac{3}{2}G_{20'}\Psi_2\tau + 3\Psi_2\tilde{\tau}' + (\text{þ} - 2\rho)\vartheta\Psi_3 - (\eth' - 3\tau')\vartheta\Psi_2, \tag{A.8r}$$

$$0 = -\tfrac{3}{2}G_{20'}\Psi_2\rho' - 3\Psi_2\tilde{\sigma}' + (\text{þ} - \rho)\vartheta\Psi_4 - (\eth' - 4\tau')\vartheta\Psi_3, \tag{A.8s}$$

and the reality conditions

$$\overline{\tilde{\epsilon}} - \overline{\tilde{\rho}} = \tilde{\epsilon} - \tilde{\rho}, \qquad \overline{\tilde{\beta}} - \overline{\tilde{\tau}} = \tilde{\beta}' - \tilde{\tau}', \qquad \overline{\tilde{\beta}'} - \overline{\tilde{\tau}'} = \tilde{\beta} - \tilde{\tau}. \tag{A.9}$$

Appendix B. Linearized parameters in ORG

B.1. **Linearized mass.** Performing a variation δM of the mass parameter of the Kerr metric in Eddington-Finkelstein coordinates yields

$$\delta g_{ab} = -\frac{4n_a n_b r}{\Sigma}\delta M, \tag{B.1}$$

which satisfies the ORG condition. Thus, we have in the Znajek tetrad, the only non-vanishing metric component is $G_{00'} = -4r\Sigma^{-1}\delta M$. The only non-vanishing components of the linearized connection, as in equation (3.4), and linearized curvature are

$$\tilde{\epsilon} = \frac{1}{9\sqrt{2}\kappa_1{}^2}\delta M, \qquad \tilde{\kappa} = \frac{i\sqrt{2}ar\sin\theta}{9\kappa_1{}^2\Sigma}\delta M, \qquad \tilde{\rho} = -\frac{\sqrt{2}r}{3\kappa_1\Sigma}\delta M, \qquad \vartheta\Psi_2 = \frac{\delta M}{27\kappa_1{}^3}. \tag{B.2}$$

The rescaled metric components are

$$\widehat{G}_2 = 0, \qquad \widehat{G}_1 = 0, \qquad \widehat{G}_0 = -\tfrac{2}{81}\delta M. \tag{B.3}$$

B.2. **Linearized angular momentum.** Performing a variation δa of the angular momentum parameter per unit mass a of the Kerr metric in Eddington-Finkelstein coordinates and transforming to ORG gauge[11] yields in the Znajek tetrad the non-vanishing components

$$G_{00'} = \frac{4Ma(1+\cos^2\theta)r}{\Sigma^2}\delta a, \qquad G_{01'} = -\frac{2iMr\sin\theta}{3\bar{\kappa}_{1'}\Sigma}\delta a, \qquad G_{10'} = \frac{2iMr\sin\theta}{3\kappa_1\Sigma}\delta a. \tag{B.4}$$

The non-vanishing components of the linearized connection and curvature are

$$\tilde{\beta} = \frac{iM\sin\theta}{6\sqrt{2}\kappa_1\Sigma}\delta a, \qquad \tilde{\beta}' = \frac{iM\sin\theta(\kappa_1 + 2\bar{\kappa}_{1'})}{6\sqrt{2}\kappa_1{}^2\Sigma}\delta a, \tag{B.5a}$$

$$\tilde{\tau} = -\frac{iMr\sin\theta}{\sqrt{2}\Sigma^2}\delta a, \qquad \tilde{\tau}' = \frac{iM\sin\theta}{27\sqrt{2}\kappa_1{}^3}\delta a, \tag{B.5b}$$

$$\tilde{\sigma} = -\frac{Mar\sin^2\theta}{3\sqrt{2}\kappa_1\Sigma^2}\delta a, \qquad \vartheta\Psi_1 = \frac{iM(a^2+r^2)\sin\theta}{486\kappa_1{}^6}\delta a, \tag{B.5c}$$

$$\vartheta\Psi_2 = \frac{M(a+ir\cos\theta)}{81\kappa_1{}^5}\delta a, \qquad \vartheta\Psi_3 = -\frac{iM\sin\theta}{54\kappa_1{}^4}\delta a. \tag{B.5d}$$

[11] The generator for the transformation is $\nu_a = -\frac{\sqrt{2}ar\sin^2\theta}{\Sigma}n_a - \frac{i}{\sqrt{2}}\sin\theta m_a + \frac{i}{\sqrt{2}}\sin\theta\bar{m}_a = -a\cos\theta\sin\theta(d\theta)_a - r\sin^2\theta(d\phi)_a$.

The rescaled metric components are

$$\widehat{G}_2 = 0, \qquad \widehat{G}_1 = -\tfrac{1}{81} i\sqrt{2} M \sin\theta \delta a, \qquad \widehat{G}_0 = \frac{2Ma(1+\cos^2\theta)}{81\Sigma}\delta a. \tag{B.6}$$

Index of symbols

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